SEQUENCES

Abbreviations

$\forall$: 'for all'
$\exists$: 'there exists'

Monotonic decreasing if $u_{n+1} \leq u_n$ and strictly decreasing if $u_{n+1} < u_n$.

Red texts : restrictions of the theorem
Bold texts : Subtitle
Italic texts : Important

Triangle Inequality

$|x+y| \leq |x|+|y|$

ε-N Definition Of Convergences

The sequence $\{u_n\}$ converges to L as $n \to \infty$ if, given any $\varepsilon > 0$ $\exists$ N such that,

$|u_n - L| < \varepsilon$ whenever $n > N$.

(| | sign shouldn't be removed until you are sure it is >0)

Some tricks for for limits

(a) $n - \alpha < n$ $\forall\ n \geq 1$;
(b) $n + \alpha < n + \alpha n$ $\forall\ n \geq 1$;
(c) $(n+\alpha)^{-1} < (n)^{-1}$ $\forall\ n \geq 1$;
(d) $(n-\alpha)^{-1} < 2(n)^{-1}$ $\forall\ n \geq 2\alpha$.

The sequence $\{u_n\}$ diverges to ∞ if given any $K > 0$ $\exists$ N such that $u_n > K$ whenever $n > N$. $u_n \to \infty$.

Asymptotic Behaviour

(a) If $u_n/v_n \to 1$ as $n \to \infty$, we say that u_n asymptotically equivalent to v_n and write $u_n \sim v_n$.
(b) If $|u_n/v_n| < K$ $\forall$ large n and constant K, we say $u_n = O(v_n)$
   a. And if $u_n = O(v_n)$ and $v_n = O(u_n)$, then we say that u_n and v_n are *comparable order*. ($u_n \asymp v_n$)
(c) If $u_n/v_n \to 0$ as $n \to \infty$, we say $u_n = o(v_n)$

Theorem (Algebra of limits)

If $u_n \to u$, $v_n \to v$, $n \to \infty$.
(a) $u_n + v_n \to u + v$;
(b) $k.u_n \to k.u$ for constant k;
(c) $u_n.v_n \to uv$;
(d) If $v_n \neq 0$, $u_n/v_n \to u/v$;
(e) $|u_n| \to 0$ IFF $u_n \to 0$;
(f) If α is a constant and $|\alpha| < 1$, then $\alpha^n \to 0$;
(g) If $\alpha > 0$, $n^{-\alpha} \to 0$.

Sandwhich Theorem

$u_n \leq v_n \leq w_n$, where $u_n \to L$ and $w_n \to L$ as $n \to \infty$, then $v_n \to L$ as $n \to \infty$

A set $S \subset \mathbb{R}$ is said to be *bounded above* if $\exists$ $x \in \mathbb{R}$ such that $y \leq x$ $\forall$ $y \in S$, and the number x is called the *supremum* of S or *the least upper bound* of S. [x = sup S]

A set $S \subset \mathbb{R}$ is said to be *bounded below* if $\exists$ $z \in \mathbb{R}$ such that $z \leq y$ $\forall$ $y \in S$, and the number z is called the *infimum* of S or *the greatest lower bound* of S. [x = sup S]

Completeness Theorem

Any non-empty set $S \subset \mathbb{R}$ that is bounded above have a *supremum*.

Any non-empty set $S \subset \mathbb{R}$ that is bounded below have an *infimum*.

Monotonic Sequence Theorem

If a sequence is bounded and behave monotically increasing/decreasing, then the sequence converges.

If a set S is not bounded above and below, then sup S = ∞ and Inf S = $-\infty$

Unnamed Theorem 1

If the superior and inferior limits both exists, then
<u>The sequence converges if,</u>
the inferior and superior limits equal

The sequence diverge by bounded oscillation if,
the inferior limits < superior limits

Subsequences

If $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $u_{f(n)}$ is a subsequence of u_n

Bolzano-Weirstrass Theorem

If a real sequence is bounded, then it contains a convergent subsequence.

Cauchy Sequences

A sequence $\{u_n\}$ is called a Cauchy sequence if given any $\varepsilon < 0$ $\exists$ N such that,
$|u_m - u_n| < \varepsilon$ whenever $m,n > N$

Cauchy Convergence Criterion

A sequence converges if and only if it is a Cauchy sequence

Contractive Sequences

If a sequence $\{u_n\}$ has the contraction property, then

$|u_{n+2} - u_{n+1}| \leq \kappa | u_{n+1} - u_n | \ \forall \ n,$

Where the contractive constant κ is a positive constant, with $\kappa < 1$ then the sequence converges

The Order Hierarchy

If $\alpha,\beta > 0$, and for brevity in this discussion we shall write $u_n \ll v_n$ if
$u_n = o(v_n)$
$1 \ll \log n \ll n^\alpha \ll e^{\beta n} \ll n! \ll n^n$

FUNCTIONS, LIMITS AND CONTINUITY

The Cartesian Product

Basically the same as the dot product of two set

General Definition

For brevity onwards, we are suppossing two sets A and B, with $a \in A$, and $b \in B$, and a function f(x) of a mapping $A \to B$.

Function of Mapping

Mapping A to B, where two different a can map to one b, however not the opposites.

Image

An image of an element a of A under f(x) is all element of B, where suffices $b = f(a)$.

Domain and Range

If $f : A \to B$, A is called domain and B is called codomain. While if the images of elements of A is called the range of f.

Example,
A : {0,1,2,3,4,5}
B : {3,4,5,6,7,8}
With $f(x) = 2x$

{0,1,2,3,4,5} is domain
{3,4,5,6,7,8} is codomain
{4,6,8} is image/range

Surjection, Injection, Bijection

Surjection

Every b is hit by one or more elements of A

Injection or one-to-one mapping

Every a got one unique image on B

Bijection or one-to-one correspondence

The combination of both surjection and injection.

Unnamed Theorem 2

A strictly monotonic function has an inverse.

Limits Of Function

If f(x) is defined $\forall$ x in an interval (a,∞), we say that $f(x) \to L$ as $x \to \infty$ if, given any $\varepsilon > 0$ $\exists$ $\zeta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ when } x > \zeta$$

If f(x) is defined $\forall$ x in an interval $(-\infty,a)$, we say that $f(x) \to L$ as $x \to -\infty$ if, given any $\varepsilon > 0$ $\exists$ $\zeta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ when } x < -\zeta$$

The algebra of limits theorem and sandwhich theorem for sequences apply as well to the function.

Limit At A Point

We say that $f(x) \to L$ as $x \to a$ if, given any $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ when } 0 < |x - a| < \delta$$

Standard Limits

For brevity in this discussion ONWARD we shall write, lim (u) is the limit as $n\to\infty$ of u, while lim0 (u) is the limit as $n\to0$ of u, lim1 (u) is the limit as $n\to1$ of u and so on.

Suppose c is constant here

$\lim (n^{-c}) = 0 \ (c > 0)$
$\lim (c^{n}) = 0 \ (|c| < 1)$
$\lim (c^{1/n}) = 0 \ (c > 0)$
$\lim (n^{1/n}) = 1$

$\lim0 (\sin(t)\, t^{-1}) = 1$
$\lim0 (\tan(t)\, t^{-1}) = 1$
$\lim0 (t \log(t^{-1})) = 0$
$\lim (\log(t)\, t^{-1}) = 0$
$\lim (1 + c\, n^{-1})^{n} = e^{c}$