FINANCIAL MATHS I

NOTES BY:
LEONARDUS EDWIN HARTONO LIMBRI

CHAPTER 1
FUNDAMENTAL CONCEPTS

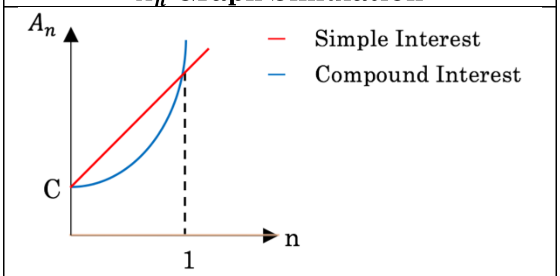
1.1 Simple and compound interest

Interest is a percentage of the principal deposit that is paid to the people who deposits their money in a bank.

→ It is an incentive for people to deposit their money in the bank
→ Encourage people to save money rather than spending it

3 things affecting the amount interest paid

- Interest rates offered by the bank (denoted as i). Usually as an annual percentage.
  i.e. 6% per annum
- Times left on deposits (denoted as n)
- The amount initially invested. Known as Principal (denoted as C)
  Principal ≠ Capital. Capital usually have a broader meaning than principal.

<table>
<tr><th colspan="2">Simple VS Compound</th></tr>
<tr><td>Simple Interest</td><td>Compound Interest</td></tr>
<tr><td colspan="2">Main differences</td></tr>
<tr><td>Simple Interest (SI) give the same amount of interest each period</td><td>Compound Interest (CI) give interest on the principal and also last period interest</td></tr>
<tr><td colspan="2">A_n, supposing A_n is the accumulated investment at time n</td></tr>
<tr><td>$A_n = C(1 + i \cdot n)$</td><td>$A_n = C \cdot (1 + i)^n$</td></tr>
<tr><td colspan="2">A_n Graph Simulation</td></tr>
<tr><td colspan="2">
</td></tr>
<tr><td colspan="2">Therefore, from the graph we see that, $A_n\ SI \geq A_n\ CI$ for $n \leq 1$, and $A_n\ SI \leq A_n\ CI$ for $n \geq 1$</td></tr>
<tr><td colspan="2">I_n, supposing I_n is the interest at time n</td></tr>
<tr><td>$I_n = C \cdot i$
Therefore, the interest paid is the same for any period n</td><td>$I_n = C[(1 + i)^n - 1]$
The interest paid takes account interest paid last period</td></tr>
</table>

Why CI is more attractive than SI
As an investment of 1 goes to SI with rate of i have A_n equivalent with A_n of investment of 1 goes to CI with rate smaller than i.

1.2 Present value and discounted value

We know that $A_n = C(1 + i)^n$, with assumptions:

1. Rate of interests does not depend on the amount invested
2. Rate of interest does not vary over the period
3. Dealing with integer number of units

Present value
The amount of money needed to have another amount of money at a future date.

Supposedly we wanted to get B amount of money after n time units, with interest rates of $i/time\ periods$. Then the amount invested (C) or the present value (PV) is

$$A_n = B = C(1 + i)^n$$
$$C = \frac{B}{(1 + i)^n}$$

Using the term $v = (1 + i)^{-1}$, which defined as the present value of 1 due at one time unit

$$\therefore \boldsymbol{PV = C = Bv^n}$$

Sometimes calculate a discounted value at some point in the future,

Assuming t and n are integers, to get D at time n, $(0 \leq t \leq n)$

$$\therefore \boldsymbol{PV = C = Dv^{n-t}}$$

"Checking progress in **x** year time means that the investment started at $t = x$ years.

1.3 Nominal and effective rates of interest

Effective rates of interest (i)
Interest per time unit i that accumulates an amount of 1 at time t to $(1 + i)$ at time $(t + 1)$

Nominal rates of interest ($i^{(m)}$)
We say that $i^{(m)}$ is the nominal rate of interest per time unit time convertible m-thly

$\therefore \frac{i^{(m)}}{m} =$ Effective rates of interest **per unit time convertible m-thly**

Important equivalents
Some important equivalents of i *and* d,

$$v = 1 - d$$

$$(1-d)(1+i) = 1$$

$$\left(1 - \frac{d^{(p)}}{p}\right)\left(1 + \frac{i^{(p)}}{p}\right) = 1$$

$$1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p$$

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p$$

1.4 Interest rates that vary over time
Effective rate of interest per time unit for period $(t, t+1)$ denoted as $i(t)$
→ Value of 1 at time t becomes $1 + i(t)$ at time $(t+1)$
→ Usually $i(t)$ is denoted as a form of function, for example $i(t) = 0.05 + 0.001t$

Nominal rate of interest $[i_h(t)]$, means that the effective rate of interest for a **period of length h**, starting at time **t** is $\boldsymbol{h \cdot i_h(t)}$**.**

Illustration
$i_{0.7}(0) = 0.08$,
This statement means that, the interest rate is **0.08** for **0.7**-time unit and the amount **1** at time **0** become $\mathbf{1 + (0.08 \cdot 0.7)}$ at time $\mathbf{(0 + 0.7)}$.

Accumulation
$A(t_1, t_2)$ is the accumulation at time t_2 of an investment **1** at time t_1

$$\therefore \boldsymbol{A(t, t+h) = 1 + h \cdot i_h(t)}$$

Rearranging formula above, we can conclude that,

$$\therefore \boldsymbol{i_h(t) = \frac{A(t, t+h) - 1}{h}}$$

Consistent market
If $t_1 < t_a < t_b < t_2$, then the following is true

$$\therefore \boldsymbol{A(t_1, t_2) = A(t_1, t_a) \cdot A(t_a, t_b) \cdot A(t_b, t_2)}$$

1.5 The force of interest
The force of interest at time t (denoted as $\delta(t)$) is the **limit** of **nominal interest rate** $\boldsymbol{i_h(t)}$ as the interval time **h tends to 0** from **above**.

$$\therefore \boldsymbol{\delta(t) = \lim_{h \to 0^+} [i_h(t)]}$$

As we know that from 1.4 that

$$i_h(t) = \frac{A(t, t+h) - 1}{h}$$

Assuming that it is in a consistent market,

$$\delta(t) = \lim_{h \to 0^+} \left[\frac{A(t, t+h) - 1}{h}\right]$$

$$\delta(t) = \frac{1}{A(0,t)} \lim_{h \to 0^+} \left[\frac{A(0, t+h) - A(0,t)}{h}\right]$$

Suppose $F(t)$ is the accumulated value at time t for investment at time 0

$$\delta(t) = \frac{1}{F(t)} \lim_{h \to 0^+} \left[\frac{F(t+h) - F(t)}{h}\right]$$

By the definition of derivatives,

$$\lim_{h \to 0^+} \left[\frac{F(t+h) - F(t)}{h}\right] = \frac{d}{dt}[F(t)] = F'(t)$$

$$\delta(t) = \frac{F'(t)}{F(t)} = \frac{d}{dr}[\log F(r)]$$

Therefore,

$$\therefore \boldsymbol{A(0,t) = F(t) = exp\left\{\int_0^t [F(r)] dr\right\}}$$

$$\therefore \boldsymbol{A(t, th) = exp\left\{\int_t^{t+h} [F(r)] dr\right\}}$$

$$\therefore \boldsymbol{1 + h \cdot i_h(t) = exp\left\{\int_t^{t+h} [F(r)] dr\right\}}$$

1.6 Present value with vary interest rates
$[A(0,t)]^{-1}$ is the PV of 1 due at time t, recall that this is the definition of v. Therefore,

$$\therefore \boldsymbol{v(t) = \exp\left\{-\int_t^{t+h} [F(r)] dr\right\}}$$

Similarly, $[A(t_1, t_2)]^{-1}$ is the PV of 1 at time t_1 due at time t_2

$$\therefore [\boldsymbol{A}(\boldsymbol{t_1}, \boldsymbol{t_2})]^{-1} = \frac{[\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{t_2})]^{-1}}{[\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{t_1})]^{-1}} = \frac{\boldsymbol{v}(\boldsymbol{t_2})}{\boldsymbol{v}(\boldsymbol{t_1})}$$

$$\therefore [\boldsymbol{A}(\boldsymbol{t_1}, \boldsymbol{t_2})]^{-1} = \boldsymbol{exp}\left\{\frac{\int_0^{t_1}[F(r)]dr}{\int_0^{t_2}[F(r)]dr}\right\}$$

1.7 Constant force of interest

For a constant force of interest, then $\delta(t)$ is some constant.

$\rightarrow v = (1+i)^{-1} = e^{-\delta}$
$\rightarrow (1+i) = e^{\delta}$

$$\therefore \boldsymbol{A}(\boldsymbol{0}, \boldsymbol{n}) = \boldsymbol{e^{\delta n}}$$

From above we could get that,

$$1 + \frac{i^{(p)}}{p} = A\left(0, \frac{1}{p}\right) = e^{\frac{\delta}{p}}$$

By algebraic manipulation,

$$\therefore \boldsymbol{i^{(p)}} = \boldsymbol{p} \cdot \left(\boldsymbol{e^{\frac{\delta}{p}}} - \boldsymbol{1}\right)$$

$$\therefore \boldsymbol{d^{(p)}} = \boldsymbol{p} \cdot \left(\boldsymbol{1} - \boldsymbol{e^{-\frac{\delta}{p}}}\right)$$

For $p > 1$ & $\delta > 0$,

$$\boldsymbol{d} < \boldsymbol{d^{(p)}} < \boldsymbol{d^{(p+1)}} < \boldsymbol{\delta} < \boldsymbol{i^{(p+1)}} < \boldsymbol{i^{(p)}} < \boldsymbol{i}$$

CHAPTER 2
VALUING CASH FLOWS

2.1 Unit amounts at unit intervals

$$a_n = v + v^2 + v^3 + \cdots + v^n$$
$$\underline{(1+i)a_n = 1 + v + v^2 + v^3 + \cdots + v^{n-1}} \; -$$

$$(-i)a_n = v^n - 1$$
$$\therefore \boldsymbol{a_n} = \frac{\boldsymbol{1} - \boldsymbol{v^n}}{\boldsymbol{i}}$$

$$\ddot{a}_n = 1 + v + v^2 + \cdots + v^{n-1}$$
$$\underline{(1-d)\ddot{a}_n = v + v^2 + v^3 + \cdots + v^n} \; -$$

$$(d)\ddot{a}_n = 1 - v^n$$
$$\therefore \boldsymbol{\ddot{a}_n} = \frac{\boldsymbol{1} - \boldsymbol{v^n}}{\boldsymbol{d}}$$

$$s_n = 1 + (1+i) + \cdots + (1+i)^{n-1}$$
$$\underline{(1+i)s_n = (1+i) + (1+i)^2 + \cdots + (1+i)^n} \; -$$

$$(-i)s_n = 1 - (1+i)^n$$

$$\therefore \boldsymbol{s_n} = \frac{(\boldsymbol{1} + \boldsymbol{i})^{\boldsymbol{n}} - \boldsymbol{1}}{\boldsymbol{i}}$$

$$\ddot{s}_n = (1+i) + (1+i)^2 + \cdots + (1+i)^n$$
$$\underline{(1-d)\ddot{s}_n = 1 + (1+i) + \cdots + (1+i)^{n-1}} \; -$$

$$(d)\ddot{s}_n = (1+i)^n - 1$$
$$\therefore \boldsymbol{\ddot{s}_n} = \frac{(\boldsymbol{1} + \boldsymbol{i})^{\boldsymbol{n}} - \boldsymbol{1}}{\boldsymbol{d}}$$

$$\therefore \boldsymbol{m|a_n} = \boldsymbol{v^m} \cdot \boldsymbol{a_n}$$

$$\therefore \boldsymbol{m|\ddot{a}_n} = \boldsymbol{v^m} \cdot \boldsymbol{\ddot{a}_n} = \boldsymbol{v^{m-1}} \cdot \boldsymbol{a_n}$$

$\rightarrow$ The interest rates here are already effective per time period.

2.2 Multiple equal payments per unit

$$a_n^{(p)} = \frac{1}{p} \cdot \left[v^{\frac{1}{p}} + v^{\frac{2}{p}} + \cdots + v^{\frac{np}{p}}\right]$$

$$a_n^{(p)} = \frac{1}{p} \cdot v^{\frac{1}{p}} \cdot \left[1 + v^{\frac{1}{p}} + v^{\frac{2}{p}} + \cdots + v^{\frac{np-1}{p}}\right]$$

$$a_n^{(p)} = \frac{1}{p} \cdot \left(1 + \frac{i^{(p)}}{p}\right)^{-1} \cdot \frac{1 - v^n}{1 - \left(1 + \frac{i^{(p)}}{p}\right)^{-1}}$$

$$\frac{\left(1 + \frac{i^{(p)}}{p}\right)^{-1}}{1 - \left(1 + \frac{i^{(p)}}{p}\right)^{-1}} \cdot \frac{\left(1 + \frac{i^{(p)}}{p}\right)}{\left(1 + \frac{i^{(p)}}{p}\right)} = \frac{1}{1 + \frac{i^{(p)}}{p} - 1} = \frac{p}{i^{(p)}}$$

$$a_n^{(p)} = \frac{1}{p} \cdot \frac{p}{i^{(p)}} \cdot (1 - v^n)$$

$$\therefore \boldsymbol{a_n^{(p)}} = \frac{\boldsymbol{1} - \boldsymbol{v^n}}{\boldsymbol{i^{(p)}}}$$

Similarly,

$$\therefore \boldsymbol{\ddot{a}_n^{(p)}} = \frac{\boldsymbol{1} - \boldsymbol{v^n}}{\boldsymbol{d^{(p)}}}$$

$$s_n^{(p)} = \frac{1}{p}\left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{np-1}{p}}\right]$$

$$s_n^{(p)} = \frac{1}{p} \cdot \frac{(1+i)^n - 1}{(1+i)^{\frac{1}{p}} - 1} = \frac{1}{p} \cdot \frac{(1+i)^n - 1}{1 + \frac{i^{(p)}}{p} - 1}$$

$$\therefore \boldsymbol{s_n^{(p)}} = \frac{(\boldsymbol{1} + \boldsymbol{i})^{\boldsymbol{n}} - \boldsymbol{1}}{\boldsymbol{i^{(p)}}}$$

Similarly,

$$\therefore \boldsymbol{\ddot{s}_n^{(p)}} = \frac{(\boldsymbol{1} + \boldsymbol{i})^{\boldsymbol{n}} - \boldsymbol{1}}{\boldsymbol{d^{(p)}}}$$