An H_∞ control design approach to networked control systems

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***Abstract*— This paper is concerned with the design problem of robust H_∞ control for linear networked control systems (NCSs) with network-induced delay and data packet dropout. By choosing a new Lyapunov-Krasovskii functional, a sufficient condition on the existence of robust H_∞ controller is derived in the form of a matrix inequality. No model transformation is needed and no redundant matrix variable is introduced. Then an iterative algorithm is introduced for obtaining the robust H_∞ controller design method based on the matrix inequality. No parameter needs to be selected in advance. Two numerical examples are finally given to illustrate the effectiveness of the proposed algorithm.**

I. INTRODUCTION

In many complicated control systems, such as manufacturing plants, power plants, automobiles, aircraft, and robot manipulators, communication networks are employed to exchange information and control signals between spatially distributed system components, like supervisory computers, controllers, and intelligent input-output (I/O) devices, e.g. smart sensors and actuators [10]. The feedback control systems wherein the control loops are closed via the communication channel are called networked control systems (NCSs). The introduction of control network "bus" architectures can improve the efficiency, flexibility, and reliability of these integrated applications, reducing installation, reconfiguration, and maintenance time and costs [6]. Since an NCS operates over a network, data transfers between the controller and the remote system, e.g. sensors and actuators in a distributed control system, will induce network delay in addition to the controller processing delay. There are essentially three kinds of delays: (1) communication delay τ^{sc} between the sensor and the controller; (2) computational delay τ^c in the controller and (3) communication delay τ^{ca} between the controller and the actuator [7]. The overall network-induced delay, which is also the transfer delay of data packets, can be computed by $\tau = \tau^{sc} + \tau^c + \tau^{ca}$. Due to many uncertain factors, the network-induced delays are generally considered as time-varying ones [9], [15], [12]. Some methodologies have been formulated based on several types of network behaviors and configurations in conjunction with different ways to treat the delay problem [7], [12]. As for H_∞ control of NCSs, few results are available in the literature. Recently, [12] has investigated the problem of robust H_∞ control of the following system controlled through a network.

$$\begin{cases} \dot{x}(t) = [A+\Delta A(t)]x(t)+[B+\Delta B(t)]u(t)+B_w w(t), \\ z(t) = Cx(t) + D_1 u(t), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^p$ is the input vector; $w(t) \in \mathcal{L}_2[0,\infty)$ is the exogenous disturbance signal; $z(t) \in \mathbb{R}^r$ is the controlled output; A, B , B_w, C, and D_1 are some constant matrices of appropriate dimensions; $\Delta A(t)$ and $\Delta B(t)$ denote the parameter uncertainties of the system (1) satisfying

$$[\ \Delta A(t) \quad \Delta B(t)\ ] = DF(t)[\ E \quad E_b\ ], \tag{2}$$

where D, E, and E_b are some constant matrices of appropriate dimensions. $F(t) \in \mathbb{R}^{l\times m}$ is an unknown time-varying matrix function with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \le I. \tag{3}$$

Both time-varying network-induced delay and data packet dropout are considered by [12] simultaneously. The following bounding was used in [12].

$$\begin{aligned} & -\int_{t-\eta}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \\ & = -\int_{i_k h}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds - \int_{t-\eta}^{i_k h} \dot{x}^T(s)R_1\dot{x}(s)ds \\ & \le -\int_{i_k h}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds. \end{aligned}$$

It is clear that $-\int_{t-\eta}^{i_k h} \dot{x}^T(s)R_1\dot{x}(s)ds \le 0$ due to $R_1 > 0$. $-\int_{t-\eta}^{i_k h} \dot{x}^T(s)R_1\dot{x}(s)ds$ was bounded by the over bounding "0". Moreover, when designing the controller, some matrix variables $M_i (i = 1,2,3,4)$, which are in fact "independent" ones, need to be supposed that $M_i = \rho_i M_1 (i = 2,3,4)$ and some parameters $\rho_i (i = 2,3,4)$ need to be selected in advance. These have brought more conservatism for controller design. How to reduce the conservatism motivates the present study.

In this paper, we will study the problem of designing robust H_∞ network-based controller for the system (1) by employing the following new functional

$$\begin{aligned} V(t) \quad = \quad & x^T(t)Px(t) + \int_{t-\tau_m}^{t} x^T(s)Q_1x(s)ds \\ & +\tau_m \int_{-\tau_m}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \\ & + \int_{t-\eta}^{t} x^T(s)Q_2x(s)ds \\ & +\eta \int_{-\eta}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \\ & +\delta \int_{-\eta}^{-\tau_m} ds \int_{t+s}^{t} \dot{x}^T(\theta)S\dot{x}(\theta)d\theta, \end{aligned} \tag{4}$$

where $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, and $S > 0$. In order to derive a much less conservative result, we will avoid the over bounding for $-\int_{t-\eta}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds$, which is corresponding to $-\int_{t-\eta}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds$ in [12]. Based on an integral inequality recently derived in [4], we will give a sufficient condition on the existence of robust

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H_∞ controller in the form of matrix inequalities. No model transformation will be needed and no redundant matrix variable will be introduced. In order to design the controller by using MATLAB LMI Toobox, we will introduce an iterative algorithm. No parameter will be selected in advance. We will also give some numerical examples to illustrate the effectiveness of the proposed results.

II. Problem Statement

In the following, for simplicity, we will write

$$A(t) \triangleq A + \Delta A(t), B(t) \triangleq B + \Delta B(t).$$

Throughout this paper, suppose that all the system's states are available for state feedback. In the presence of the control network, which is shown in Fig. 1, data transfers between the controller and the remote system, e.g. sensors and actuators in a distributed control system will induce networked delay in addition to the controller proceeding delay. There are essentially three kinds of delay: (1) communication delay τ^{sc} between the sensor and the controller; (2) computational delay τ^c in the controller; and (3) communication delay τ^{ca} between the controller and the actuator.

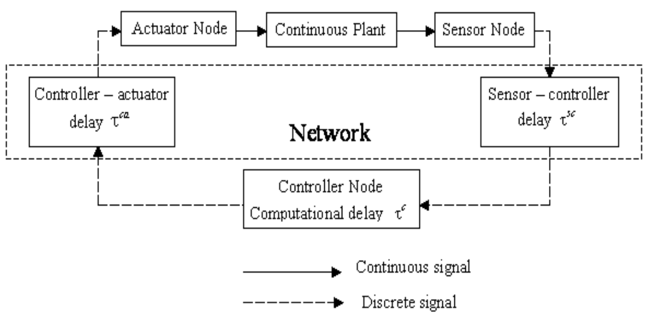

Fig. 1. The networked control system.

First, since there exists the communication delay τ^{sc} between the sensor and the controller and computational delay τ^c in the controller, which is shown in Fig. 1, the following control law is employed for the system (1) – (3)

$$\begin{aligned} u(t^+) = Kx(t - \tau_k^{sc} - \tau_k^c), \\ t \in \{kh + \tau_k^{sc} + \tau_k^c, k = 1, 2, \cdots\}, \end{aligned} \tag{5}$$

where K is a controller gain to be determined. h is the sampling period. Second, substituting (5) into (1) yields the closed-loop system by considering the communication delay τ^{ca} between the controller and the actuator

$$\left\{ \begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)Kx(kh) + B_w w(t), \\ &z(t) = Cx(t) + D_1 Kx(kh), \\ &\qquad t \in [kh + \tau_k, (k+1)h + \tau_{k+1}),\ k = 1, 2, \cdots. \end{aligned} \right. \tag{6}$$

Considering the data packet dropout, the closed-loop system (6) can be modified as [12]

$$\left\{ \begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)Kx(i_k h) + B_w w(t), \\ &z(t) = Cx(t) + D_1 Kx(i_k h), \\ &\qquad t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1}),\ k = 1, 2, \cdots, \end{aligned} \right. \tag{7}$$

where i_k, $k = 1, 2, \cdots$ are some integers and $\{i_1, i_2, i_3, \cdots\} \subset \{0, 1, 2, \cdots\}$. The time-delay $\tau_k = \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$ denotes the time from the instant $i_k h$ when sensor nodes sample sensor data from a plant to the instant when actuators transfer data to the plant (as shown in Fig. 2). Obviously, $\bigcup_{k=1}^{\infty} [i_k h + \tau_k, i_{k+1} h + \tau_{k+1}) = [t_0, \infty)$, $t_0 \geq 0$. In this paper, $u(t) = 0$ is assumed before the first control signal reaches the plant.

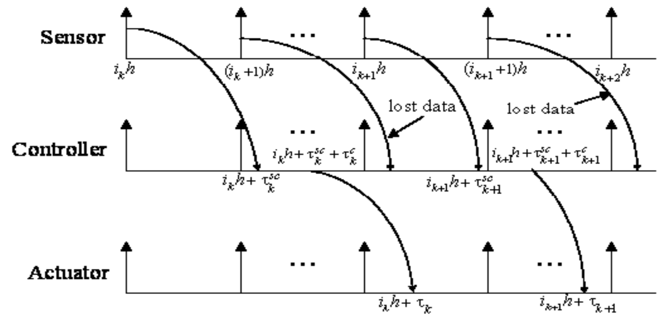

Fig. 2. Time diagram for data packets.

Throughout this paper, the following assumptions, definition and lemma are needed.

Assumption 1: [12] The sensor is clock-driven; the controller and actuator are event-driven.

Assumption 2: [12] There exist two constants $\tau_m \geq 0$ and $\eta > 0$ such that

$$\left\{ \begin{aligned} &(i_{k+1} - i_k)h + \tau_{k+1} \leq \eta, \\ &\tau_k \geq \tau_m,\ k = 1, 2, \cdots \end{aligned} \right. \tag{8}$$

Remark 1: Since $x(i_k h) = x(t - (t - i_k h))$, defining $\tau(t) = t - i_k h$, $t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1})$, $k = 1,2,\cdots$, rewrite (7) as

$$\left\{ \begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)Kx(t - \tau(t)) + B_w w(t), \\ &z(t) = Cx(t) + D_1 Kx(t - \tau(t)), \end{aligned} \right. \tag{9}$$

where $\tau(t)$ is piecewise-linear with derivative $\dot{\tau}(t) = 1$ for $t \neq t_k$ and $\tau(t)$ is discontinuous at the points $t = t_k$, $k = 1,2,\cdots$. It is clear that $\tau_k \leq \tau(t) \leq (i_{k+1} - i_k)h + \tau_{k+1}$ for $t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1})$, $k = 1,2,\cdots$. So the system (7) is equivalent to the linear system (9) with interval time-varying delay. The initial condition of the state $x(t)$ on $[t_0 - \eta, t_0]$ is supplemented as

$$x(t) = \phi(t), t \in [t_0 - \eta, t_0] \text{ with } \phi(t_0) = x_0, \tag{10}$$

where $\phi(t)$ is a continuous function on $[t_0 - \eta, t_0]$.

Definition 1: [12] For a prescribed scalar $\gamma > 0$, system (9) is said to be robustly exponentially stable with an H_∞ norm bound γ, if

(1) System (9) with $w(t) \equiv 0$ is robustly exponentially stable, that is, there exist constants $\alpha > 0$ and $\beta > 0$ such that $\|x(t)\| \leq \alpha \sup_{t_0 - \eta \leq s \leq t_0} \|\phi(s)\| e^{-\beta(t - t_0)}$ for $t \geq t_0$ for all admissible uncertainties satisfying (2), (3);

(2) The controlled output $z(t)$ satisfies $\|z(t)\|_2 \leq \gamma \|w(t)\|_2$ for all nonzero $w(t) \in \mathcal{L}_2[0, \infty)$ under the condition $x(t) \equiv 0$, $\forall t \in [t_0 - \eta, t_0]$.

By Remark 1, the system (7) is equivalent to the time-delay system (9). In the following, what the system (7) is robustly exponentially stable with an H_∞ norm bound γ means that the delay system (9) is robustly exponentially stable with an H_∞ norm bound γ.

Lemma 1: [4] For any constant matrix $W \in \mathbb{R}^{n \times n}$, $W = W^T > 0$, scalar $r > 0$, and vector function $\dot{x} : [-r, 0] \to \mathbb{R}^n$ such that the following integration is well defined, then

$$-r\int_{-r}^{0}\dot{x}^T(t+\xi)W\dot{x}(t+\xi)d\xi \leq -(x^T(t)-x^T(t-r))W(x(t)-x(t-r)). \quad (11)$$

For a prescribed scalar $\gamma > 0$, we define the performance index

$$J(w)=\int_0^{\infty}[z^T(\theta)z(\theta)-\gamma^2 w^T(\theta)w(\theta)]d\theta. \quad (12)$$

III. MAIN RESULT

Rewrite the system described by (7), (2), and (3) as [2], [3]

$$\begin{aligned}\dot{x}(t) &= Ax(t)+BKx(i_k h)+Dv(t)+B_w w(t), && (13)\\ g(t) &= Ex(t)+E_b Kx(i_k h), && (14)\\ & t\in[i_k h+\tau_k, i_{k+1}h+\tau_{k+1}),\ k=1,2,\cdots. \end{aligned}$$

subject to uncertain feedback

$$v(t)=F(t)g(t). \quad (15)$$

In view of (3), (14), and (15), we have

$$\begin{aligned}&v^T(t)v(t)\\ &\leq [Ex(t)+E_bKx(i_kh)]^T[Ex(t)+E_bKx(i_kh)], \quad (16)\\ &t\in[i_k h+\tau_k, i_{k+1}h+\tau_{k+1}),\ k=1,2,\cdots.\end{aligned}$$

We now state and establish the following proposition which gives a sufficient condition on the existence of the network-based H_∞ controller for the system (1) – (3).

Proposition 1: For some given scalars τ_m, η, and γ, the closed-loop system (7) is robustly exponentially stable with an H_∞ norm bound γ, if there exist a scalar $\tilde{\varepsilon}>0$, matrices $X>0$, $\tilde{Q}_1>0$, $\tilde{Q}_2>0$, $\tilde{R}_1>0$, $\tilde{R}_2>0$, $\tilde{S}>0$, Y of appropriate dimensions such that

$$\Gamma=\begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{12}^T & \Gamma_{22}\end{bmatrix}<0, \quad (17)$$

where

$$\Gamma_{11}=\begin{bmatrix}\Upsilon_1 & BY & \tilde{R}_1 & \tilde{R}_2 & \tilde{\varepsilon}D & B_w\\ (BY)^T & -2\tilde{S} & \tilde{S} & \tilde{S} & 0 & 0\\ \tilde{R}_1 & \tilde{S} & \Upsilon_2 & 0 & 0 & 0\\ \tilde{R}_2 & \tilde{S} & 0 & \Upsilon_3 & 0 & 0\\ \tilde{\varepsilon}D^T & 0 & 0 & 0 & -\tilde{\varepsilon}I & 0\\ B_w^T & 0 & 0 & 0 & 0 & -\gamma^2 I\end{bmatrix},$$

$$\Gamma_{12}^T=\begin{bmatrix}\tau_m AX & \tau_m BY & 0 & 0 & \tilde{\varepsilon}\tau_m D & \tau_m B_w\\ \eta AX & \eta BY & 0 & 0 & \tilde{\varepsilon}\eta D & \eta B_w\\ \delta AX & \delta BY & 0 & 0 & \tilde{\varepsilon}\delta D & \delta B_w\\ EX & E_bY & 0 & 0 & 0 & 0\\ CX & D_1Y & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Gamma_{22}=-diag\left\{X\tilde{R}_1^{-1}X, X\tilde{R}_2^{-1}X, X\tilde{S}^{-1}X, \tilde{\varepsilon}I, I\right\},$$

with

$\delta=\eta-\tau_m, \Upsilon_1=AX+XA^T+\tilde{Q}_1+\tilde{Q}_2-\tilde{R}_1-\tilde{R}_2,$
$\Upsilon_2=-\tilde{Q}_1-\tilde{R}_1-\tilde{S}, \Upsilon_3=-\tilde{Q}_2-\tilde{R}_2-\tilde{S}.$

Moreover, the controller gain of (5) is $K=YX^{-1}$.

Proof: Taking the derivative of $V(t)$ with respect to t along the trajectory of the system (13) yields

$$\begin{aligned}\dot{V}(t) = {} & 2x^T(t)P[Ax(t)+BKx(i_kh)+Dv(t)+B_ww(t)]\\ & +x^T(t)(Q_1+Q_2)x(t)-x^T(t-\tau_m)Q_1x(t-\tau_m)\\ & -x^T(t-\eta)Q_2x(t-\eta)+\dot{x}^T(t)(\tau_m^2R_1+\eta^2R_2\\ & +\delta^2S)\dot{x}(t)-\tau_m\int_{t-\tau_m}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\\ & -\eta\int_{t-\eta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds-\delta\int_{t-\eta}^{t-\tau_m}\dot{x}^T(s)S\dot{x}(s)ds \end{aligned} \quad (18)$$

for $t\in[i_kh+\tau_k, i_{k+1}h+\tau_{k+1})$, $k=1,2,\cdots$.

Use Lemma 1 to obtain

$$\begin{aligned}&-\tau_m\int_{t-\tau_m}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\\ &\leq -[x(t)-x(t-\tau_m)]^TR_1[x(t)-x(t-\tau_m)], \quad (19)\\ &-\eta\int_{t-\eta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds\\ &\leq -[x(t)-x(t-\eta)]^TR_2[x(t)-x(t-\eta)], \quad (20)\end{aligned}$$

and

$$\begin{aligned}&-\delta\int_{t-\eta}^{t-\tau_m}\dot{x}^T(s)S\dot{x}(s)ds\\ = {} & -\delta\int_{i_kh}^{t-\tau_m}\dot{x}^T(s)S\dot{x}(s)ds-\delta\int_{t-\eta}^{i_kh}\dot{x}^T(s)S\dot{x}(s)ds\\ \leq {} & -[x(t-\tau_m)-x(i_kh)]^TS[x(t-\tau_m)-x(i_kh)]\\ & -[x(t-\eta)-x(i_kh)]^TS[x(t-\eta)-x(i_kh)]. \quad (21)\end{aligned}$$

From (18) — (21), we have

$$\begin{aligned}\dot{V}(t) \leq {} & x^T(t)[PA+A^TP+Q_1+Q_2]x(t)\\ & +2x^T(t)P[BKx(i_kh)+Dv(t)+B_ww(t)]\\ & -x^T(t-\tau_m)Q_1x(t-\tau_m)-x^T(t-\eta)Q_2x(t-\eta)\\ & +\dot{x}^T(t)(\tau_m^2R_1+\eta^2R_2+\delta^2S)\dot{x}(t)\\ & -[x(t)-x(t-\tau_m)]^TR_1[x(t)-x(t-\tau_m)]\\ & -[x(t)-x(t-\eta)]^TR_2[x(t)-x(t-\eta)]\\ & -[x(t-\tau_m)-x(i_kh)]^TS[x(t-\tau_m)-x(i_kh)]\\ & -[x(t-\eta)-x(i_kh)]^TS[x(t-\eta)-x(i_kh)]\\ = {} & \xi^T(t)\Omega\xi(t) \end{aligned} \quad (22)$$

for $t\in[i_kh+\tau_k, i_{k+1}h+\tau_{k+1})$, $k=1,2,\cdots$, where

$$\xi^T(t)=[\,x^T(t)\ \ x^T(i_kh)\ \ x^T(t-\tau_m)\ \ x^T(t-\eta)\ \ v^T(t)\ \ w^T(t)],$$

$$\Omega=\begin{bmatrix}\tilde{\Upsilon}_1 & \Omega_{12} & R_1 & R_2 & \Omega_{15} & \Omega_{16}\\ \Omega_{12}^T & \Omega_{22} & S & S & \Omega_{25} & \Omega_{26}\\ R_1 & S & \tilde{\Upsilon}_2 & 0 & 0 & 0\\ R_2 & S & 0 & \tilde{\Upsilon}_3 & 0 & 0\\ \Omega_{15}^T & \Omega_{25}^T & 0 & 0 & D^T\Theta D & D^T\Theta B_w\\ \Omega_{16}^T & \Omega_{26}^T & 0 & 0 & B_w^T\Theta D & B_w^T\Theta B_w\end{bmatrix},$$

with

$$\begin{aligned}\tilde{\Upsilon}_1 &= PA+A^TP+Q_1+Q_2-R_1-R_2+A^T\Theta A,\\ \tilde{\Upsilon}_2 &= -Q_1-R_1-S,\\ \tilde{\Upsilon}_3 &= -Q_2-R_2-S,\\ \Omega_{12} &= PBK+A^T\Theta BK,\\ \Omega_{15} &= PD+A^T\Theta D,\\ \Omega_{16} &= PB_w+A^T\Theta B_w,\\ \Omega_{22} &= (BK)^T\Theta BK-2S,\\ \Omega_{25} &= (BK)^T\Theta D\\ \Omega_{26} &= (BK)^T\Theta B_w\\ \Theta &= \tau_m^2R_1+\eta^2R_2+\delta^2S.\end{aligned}$$

From (16), using $\mathcal{S}$-procedure yields

$$\dot{V}(t)\leq\xi^T(t)\Phi\xi(t) \quad (23)$$

for $\varepsilon > 0$ and $t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1})$, $k = 1, 2, \cdots$, where

$$\Phi = \Omega + \begin{bmatrix} (1,1) & (1,2) & 0 & 0 & 0 & 0 \\ (1,2)' & (2,2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} (1,1) &= \varepsilon E^T E, \\ (1,2) &= \varepsilon E^T E_b K, \\ (2,2) &= \varepsilon (E_b K)^T E_b K. \end{aligned}$$

It is easy to see that if

$$\Phi + \begin{bmatrix} C^T C & (1,2) & 0 & 0 & 0 & 0 \\ (1,2)' & (2,2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{24}$$

$$\begin{aligned} (1,2) &= C^T D_1 K, \\ (2,2) &= (D_1 K)^T D_1 K. \end{aligned}$$

then

$$\dot{V}(t) \le -z^T(t) z(t) + \gamma^2 w^T(t) w(t) \tag{25}$$

for $t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1})$, $k = 1, 2, \cdots$. By Schur complement, (24) is equivalent to

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0 \tag{26}$$

where

$$\Phi_{11} = \begin{bmatrix} \hat{\Upsilon}_1 & PBK & R_1 & R_2 & PD & PB_w \\ (BK)^T P & -2S & S & S & 0 & 0 \\ R_1 & S & \tilde{\Upsilon}_2 & 0 & 0 & 0 \\ R_2 & S & 0 & \tilde{\Upsilon}_3 & 0 & 0 \\ D^T P & 0 & 0 & 0 & -\varepsilon I & 0 \\ B_w^T P & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$\Phi_{12}^T = \begin{bmatrix} \tau_m A & \tau_m BK & 0 & 0 & \tau_m D & \tau_m B_w \\ \eta A & \eta BK & 0 & 0 & \eta D & \eta B_w \\ \delta A & \delta BK & 0 & 0 & \delta D & \delta B_w \\ E & E_b K & 0 & 0 & 0 & 0 \\ C & D_1 K & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_{22} = diag\{-R_1^{-1}, -R_2^{-1}, -S^{-1}, -\varepsilon^{-1} I, -I\}$$

with $\hat{\Upsilon}_1 = PA + A^T P + Q_1 + Q_2 - R_1 - R_2$. Pre- and post-multiplying both sides of (26) with

$$diag\{X, X, X, X, \tilde{\varepsilon} I, I, I, I, I, I, I\}$$

and its transpose, respectively, and introducing $X = P^{-1}$, $Y = KX$, $\tilde{Q}_1 = XQ_1X$, $\tilde{Q}_2 = XQ_2X$, $\tilde{R}_1 = XR_1X$, $\tilde{R}_2 = XR_2X$, $\tilde{S} = XSX$ and $\tilde{\varepsilon} = \varepsilon^{-1}$, then using Schur complement yields (17).

First, we consider the robust exponential stability of the closed-loop system (7) with $w(t) \equiv 0$. It is clear that the following matrix inequality can be implied by (24).

$$\begin{bmatrix} (1,1) & (1,2) & R_1 & R_2 & (1,5) \\ (1,2)^T & (2,2) & S & S & (2,5) \\ R_1 & S & \tilde{\Upsilon}_2 & 0 & 0 \\ R_2 & S & 0 & \tilde{\Upsilon}_3 & 0 \\ (1,5)^T & (2,5)^T & 0 & 0 & (5,5) \end{bmatrix} < 0 \tag{27}$$

where

$$\begin{aligned} (1,1) &= \tilde{\Upsilon}_1 + \varepsilon E^T E, \\ (1,2) &= PBK + A^T \Theta BK + \varepsilon E^T E_b K, \\ (1,5) &= PD + A^T \Theta D, \\ (2,2) &= (BK)^T \Theta BK - 2S + \varepsilon (E_b K)^T E_b K, \\ (2,5) &= (BK)^T \Theta D, \\ (5,5) &= D^T \Theta D - \varepsilon I. \end{aligned}$$

From (23) there exists a $\lambda > 0$ such that

$$\dot{V}(t) < -\lambda x^T(t) x(t) - \lambda x^T(i_k h) x(i_k h)$$

for $t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1})$, $k = 1, 2, \cdots$. Defining a new functional as

$$\tilde{V}(t) = e^{\sigma t} V(t), \tag{28}$$

where $\sigma > 0$ is a constant to be determined, then we have

$$\|x(t)\|^2 \le \frac{M_1 + M_2}{\lambda_{\min}(P)} \sup_{t_0 - \eta \le s \le t_0} \|\phi(s)\|^2 e^{-\sigma(t - t_0)}, \tag{29}$$

where

$$\begin{aligned} M_1 &= \tau_m \lambda_{\max}(Q_1) e^{\sigma \tau_m} + \eta \lambda_{\max}(Q_2) e^{\sigma \eta} \\ &\quad + M_3(\|A\| + \|D\| \cdot \|E\|), \\ M_2 &= \lambda_{\max}(P) + \tau_m \lambda_{\max}(Q_1) + \eta \lambda_{\max}(Q_2) \\ &\quad + \frac{1}{2} \left(\tau_m^3 \lambda_{\max}(R_1) + \eta^3 \lambda_{\max}(R_2) \right. \\ &\quad \left. + 2\delta^2 \tau_m \lambda_{\max}(S) \right) (\|A\| + \|D\| \cdot \|E\|) \end{aligned}$$

with $M_3 = \tau_m^3 \lambda_{\max}(R_1) e^{\sigma \tau_m} + \eta^3 \lambda_{\max}(R_2) e^{\sigma \eta} + \delta^2 \eta \lambda_{\max}(S) e^{\sigma \eta}$, and $\sigma > 0$ is a sufficiently small constant such that

$$\begin{cases} -\lambda + \sigma \lambda_{\max}(P) + \sigma M_1 < 0, \\ -\lambda + \sigma M_3 (\|B\| + \|D\| \cdot \|E_b\|) \|K\| < 0. \end{cases} \tag{30}$$

From (29) and by Definition 1, we can conclude that the closed-loop system (7) with $w(t) \equiv 0$ is robustly exponentially stable if matrix inequality (26) is satisfied.

Next, we consider the performance index (12) of the system (7) under the condition $x(t) \equiv 0$, $\forall t \in [t_0 - \eta, t_0]$. Integrating both sides of (25) from $i_k h + \tau_k$ to t, where $t \in [i_k h + \tau_k, i_{k+1} h + \tau_{k+1})$, $k = 1, 2, \cdots$, we have

$$\begin{aligned} & V(t) - V(i_k h + \tau_k) \\ \le & \int_{i_k h + \tau_k}^{t} \left(-z^T(s) z(s) + \gamma^2 w^T(s) w(s) \right) ds. \end{aligned} \tag{31}$$

Since $\bigcup_{k=1}^{\infty} [i_k h + \tau_k, i_{k+1} h + \tau_{k+1}) = [t_0, \infty)$ and $V(t)$ is continuous in t, it follows from (31) that

$$V(t) - V(t_0) \le \int_{t_0}^{t} \left(-z^T(s) z(s) + \gamma^2 w^T(s) w(s) \right) ds. \tag{32}$$

Let $t \longrightarrow \infty$, then

$$\int_{t_0}^{\infty} z^T(s)z(s)ds \le \gamma^2 \int_{t_0}^{\infty} w^T(s)w(s)ds,$$

which means that $\|z(t)\|_2 \le \gamma \|w(t)\|_2$. Q.E.D.

Remark 2: As a by-product, if we do not consider the external disturbance, from the proof process of Proposition 1, we can conclude that the closed-loop system (7) with $w(t) \equiv 0$ is robustly exponentially stable if there exist a scalar $\varepsilon > 0$, matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, $S > 0$ of appropriate dimensions such that (27) is satisfied for a given controller gain K.

Remark 3: As mentioned in the Introduction, in order to derive a less conservative result, the over bounding for the term $-\int_{t-\eta}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds$ should be avoided. From (21), one can clearly see that in estimating the term $-\int_{t-\eta}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds$, instead of using the over bounding "0" for the term $-\int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)S\dot{x}(s)ds$, we use the "tighter" bounding $\begin{bmatrix} x(t-\tau(t)) \\ x(t-\tau_m) \end{bmatrix}^T \begin{bmatrix} -S & S \\ S & -S \end{bmatrix} \begin{bmatrix} x(t-\tau(t)) \\ x(t-\tau_m) \end{bmatrix}$, which is semi-negative definite due to $S > 0$.

Notice that (17) is not an LMI and it can not be solved directly by MATLAB LMI Toolbox. Similar to [13], the matrix inequality (17) can also be solved by introducing some new variables. First, we need to define three new variables T_1, T_2 and T_3 such that

$$X\tilde{R}_1^{-1}X \ge T_1, X\tilde{R}_2^{-1}X \ge T_2, X\tilde{S}^{-1}X \ge T_3. \quad (33)$$

(33) is equivalent to

$$X^{-1}\tilde{R}_1X^{-1} \le T_1^{-1}, X^{-1}\tilde{R}_2X^{-1} \le T_2^{-1}, X^{-1}\tilde{S}X^{-1} \le T_3^{-1} \quad (34)$$

or

$$\begin{cases} \begin{bmatrix} -T_1^{-1} & X^{-1} \\ X^{-1} & -\tilde{R}_1^{-1} \end{bmatrix} \le 0, \\ \begin{bmatrix} -T_2^{-1} & X^{-1} \\ X^{-1} & -\tilde{R}_2^{-1} \end{bmatrix} \le 0, \\ \begin{bmatrix} -T_3^{-1} & X^{-1} \\ X^{-1} & -\tilde{S}^{-1} \end{bmatrix} \le 0 \end{cases} \quad (35)$$

by Schur complement. Then, by introducing some new variables $\bar{T}_1$, $\bar{T}_2$, $\bar{T}_3$, $\bar{X}$, $\bar{R}_1$, $\bar{R}_2$, $\bar{S}$, (35) can be represented as

$$\begin{cases} \begin{bmatrix} -\bar{T}_1 & \bar{X} \\ \bar{X} & -\bar{R}_1 \end{bmatrix} \le 0, \begin{bmatrix} -\bar{T}_2 & \bar{X} \\ \bar{X} & -\bar{R}_2 \end{bmatrix} \le 0, \\ \begin{bmatrix} -\bar{T}_3 & \bar{X} \\ \bar{X} & -\bar{S} \end{bmatrix} \le 0, \bar{T}_i = T_i^{-1} (i = 1, 2, 3), \\ \bar{R}_j = \tilde{R}_j^{-1} (j = 1, 2), \bar{S} = \tilde{S}^{-1}, \bar{X} = X^{-1}. \end{cases} \quad (36)$$

Now, using the similar idea to a cone complementary linearization algorithm in [1], the original condition (17) can be solved by the following minimization problem involving LMI conditions instead of the original nonlinear matrix inequality.

Minimize $tr(\bar{X}X + \bar{T}_1T_1 + \bar{T}_2T_2 + \bar{T}_3T_3$ $+\bar{R}_1\tilde{R}_1 + \bar{R}_2\tilde{R}_2 + \bar{S}\tilde{S})$ subject to

$$\begin{cases} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \tilde{\Gamma}_{22} \end{bmatrix} < 0, \begin{bmatrix} -\bar{T}_1 & \bar{X} \\ \bar{X} & -\bar{R}_1 \end{bmatrix} \le 0, \\ \begin{bmatrix} -\bar{T}_2 & \bar{X} \\ \bar{X} & -\bar{R}_2 \end{bmatrix} \le 0, \begin{bmatrix} -\bar{T}_3 & \bar{X} \\ \bar{X} & -\bar{S} \end{bmatrix} \le 0, \\ \begin{bmatrix} X & I \\ I & \bar{X} \end{bmatrix} \ge 0, \begin{bmatrix} T_1 & I \\ I & \bar{T}_1 \end{bmatrix} \ge 0, \\ \begin{bmatrix} T_2 & I \\ I & \bar{T}_2 \end{bmatrix} \ge 0, \begin{bmatrix} T_3 & I \\ I & \bar{T}_3 \end{bmatrix} \ge 0, \\ \begin{bmatrix} \bar{R}_1 & I \\ I & \tilde{R}_1 \end{bmatrix} \ge 0, \begin{bmatrix} \bar{R}_2 & I \\ I & \tilde{R}_2 \end{bmatrix} \ge 0, \begin{bmatrix} \bar{S} & I \\ I & \tilde{S} \end{bmatrix} \ge 0, \end{cases} \quad (37)$$

where $\tilde{\Gamma}_{22} = -diag\{T_1, T_2, T_3, \tilde{\varepsilon}I, I\}$. Similar to [1], the above minimization problem can be solved by following iterative algorithm.

Algorithm 1:

(1) For three given constants $\tau_m \ge 0, \eta \ge 0$ and $\gamma \ge 0$, find a feasible solution under the LMIs conditions in (37)
$$(X_0, \bar{X}_0, \tilde{Q}_{10}, \tilde{Q}_{20}, \tilde{R}_{10}, \tilde{R}_{20}, \tilde{S}_0, \bar{R}_{10}, \bar{R}_{20}, \bar{S}_0, T_{10}, T_{20}, T_{30}, \bar{T}_{10}, \bar{T}_{20}, \bar{T}_{30}, \tilde{\varepsilon}_0, Y_0).$$
Set $k = 0$. If there are none, exit.

(2) Solve the following LMIs problem with a feasible solution $(X, \bar{X}, \tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2, \tilde{S}, \bar{R}_1, \bar{R}_2, \bar{S}, T_1, T_2, T_3, \bar{T}_1, \bar{T}_2, \bar{T}_3, \tilde{\varepsilon}, Y)$
Minimize $tr(\bar{X}_kX + X_k\bar{X} + \bar{T}_{1k}T_1 + T_{1k}\bar{T}_1 + \bar{T}_{2k}T_2 + T_{2k}\bar{T}_2 + \bar{T}_{3k}T_3 + T_{3k}\bar{T}_3 + \bar{R}_{1k}\tilde{R}_1 + \tilde{R}_{1k}\bar{R}_1 + \bar{R}_{2k}\tilde{R}_2 + \tilde{R}_{2k}\bar{R}_2 + \bar{S}_k\tilde{S} + \tilde{S}_k\bar{S})$
subject to LMIs in (37).
Set $X_{k+1} = X, \bar{X}_{k+1} = \bar{X}, T_{1,k+1} = T_1, T_{2,k+1} = T_2, T_{3,k+1} = T_3, \bar{T}_{1,k+1} = \bar{T}_1, \bar{T}_{2,k+1} = \bar{T}_2, \bar{T}_{3,k+1} = \bar{T}_3, \tilde{R}_{1,k+1} = \tilde{R}_1, \tilde{R}_{2,k+1} = \tilde{R}_2, \tilde{S}_{k+1} = \tilde{S}, \bar{R}_{1,k+1} = \bar{R}_1, \bar{R}_{2,k+1} = \bar{R}_2, \bar{S}_{k+1} = \bar{S}, \tilde{\varepsilon}_{k+1} = \tilde{\varepsilon}, Y_{k+1} = Y$.

(3) If the conditions (17) is satisfied or is not satisfied within a specified number of iterations, then exit. Otherwise, set $k = k + 1$ and return to Step 2).

Once a solution of matrix inequality (17) can be found by the iterative algorithm, the controller gain of (5) is designed as $K = YX^{-1}$.

Remark 4: The first step of the algorithm and every Step 2) are simple LMI problems. They can be solved by MATLAB LMI Toolbox. Similar to Theorem 2.1 in [1], the sequence $t_k \triangleq tr(\bar{X}_kX + X_k\bar{X} + \bar{T}_{1k}T_1 + T_{1k}\bar{T}_1 + \bar{T}_{2k}T_2 + T_{2k}\bar{T}_2 + \bar{T}_{3k}T_3 + T_{3k}\bar{T}_3 + \bar{R}_{1k}\tilde{R}_1 + \tilde{R}_{1k}\bar{R}_1 + \bar{R}_{2k}\tilde{R}_2 + \tilde{R}_{2k}\bar{R}_2 + \bar{S}_k\tilde{S} + \tilde{S}_k\bar{S})$, $k \ge 0$ is bounded below by $14n$ and decreasing. Thus, the sequence $\{t_k\}$ converges to some value $t_{opt} \ge 14n$. Equality holds if and only if $\bar{X}X = I$, $\bar{T}_iT_i = I (i = 1, 2, 3)$, $\bar{R}_j\tilde{R}_j = I (j = 1, 2)$ and $\bar{S}\tilde{S} = I$ at the optimum.

IV. Numerical Examples

Example 1: Consider the following system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t). \quad (38)$$

The network-based controller is designed as (5) with $K = \begin{bmatrix} -3.75 & -11.5 \end{bmatrix}$. From [14], [8], [5], [11], [12], the maximum allowable transfer delays bound is 4.5×10^{-4}, 0.0538, 0.7805, 0.8695 and 0.8871, respectively to guarantee the stability of the system (38) controlled over a network. By solving matrix inequality (27), the maximum allowable transfer delay bound is 1.0081. For this example, our result is less conservative than the results in [14], [8], [5], [11], [12].

Moreover, in order to consider the effect of the external disturbance on the system, (38) is re-expressed as [12]

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0.1u(t). \end{cases} \tag{39}$$

For the case of $\tau_m = 0$, $\gamma_{\min} = 6.82$ is found for $\eta = 0.8695$ in [12], while $\gamma_{\min} = 1.0005$ is found by solving matrix inequality (24).

Example 2: Consider the following uncertain system controlled over a network [12]

$$\begin{cases} \dot{x}(t) = \left(\begin{bmatrix} -1 & 0 & -0.5 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} + \Delta A(t) \right) x(t) \\ \qquad + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T u(t) + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T w(t), \\ z(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t) + 0.1u(t), \end{cases} \tag{40}$$

where $\|\Delta A(t)\| \leq 0.01$. In [12], $\gamma_{\min} = 1.9$ is found under the controller $u(t) = [-0.5425 \ -0.0014 \ -1.3858]x(t)$ for $\eta = 0.5$ and $\tau_m = 0.1$. However, by solving matrix inequality (24), $\gamma_{\min} = 1.6242$ is found under the same situation. By using the iterative algorithm in this paper, $\gamma_{\min} = 1.62$ is obtained under the controller $u(t) = [-0.6085 \ -0.0072 \ -1.4456]x(t)$. Therefore, the result in this paper is less conservative than that in [12] for this example.

V. Conclusions

The problem of robust H_∞ controller for linear networked control systems has been investigated. Based on a new Lyapunov-krasovskii functional, a sufficient condition on the existence of the controller has been derived in the form of a matrix inequality. No model transformation has been needed and no redundant matrix variable has been introduced. In order to obtain a solution of the obtained matrix inequality, an iterative algorithm has been introduced. No parameter has been selected in advance. The effectiveness of the proposed iterative algorithm has been illustrated through two numerical examples.

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