VI. Conclusion

We have expanded the recently proposed NNRC framework by incorporating an extra module, which is responsible for achieving fairness in allocating network resources among the competing sources. This has been satisfied by introducing a novel algorithm that controls in a stable and adaptive manner the number of communication channels in each source. Besides fairness, the ENNRC framework still guarantees small enough queues at equilibrium, preventing large queueing delays and congestion collapse, though not zero, thus avoiding link starvation. Simulations have been conducted to highlight the performance of the proposed scheme and to compare it with other well-established congestion control mechanisms. A limitation of the proposed theory seems to be the assumed availability of the ECN bit. Possible modifications in the direction of relaxing such an assumption and thus broadening the applicability of the presented methodology are currently under investigation.

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New Lyapunov–Krasovskii Functionals for Global Asymptotic Stability of Delayed Neural Networks

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***Abstract*—This brief deals with the problem of global asymptotic stability for a class of delayed neural networks. Some new Lyapunov–Krasovskii functionals are constructed by *nonuniformly* dividing the delay interval into multiple segments, and choosing proper functionals with different weighting matrices corresponding to different segments in the Lyapunov–Krasovskii functionals. Then using these new Lyapunov–Krasovskii functionals, some new delay-dependent criteria for global asymptotic stability are derived for delayed neural networks, where both constant time delays and time-varying delays are treated. These criteria are much less conservative than some existing results, which is shown through a numerical example.**

***Index Terms*—Linear matrix inequality (LMI), Lyapunov–Krasovskii functional, neural networks, stability, time delay.**

I. Introduction

Neural networks have been extensively investigated in the last two decades due to their important applications in different fields such as image processing, pattern recognition, associate memory, combinatorial optimization, solving nonlinear algebraic equations, and so on. As is well known, most of these applications depend on their stability behavior of the neural networks. However, time delays, which are unavoidable in neural networks, may induce instability of the neural networks. Therefore, the stability analysis of delayed neural networks has been received much attention in recent years. By employing Lyapunov–Kraskovskii stability theorem incorporating with a linear matrix inequality (LMI) technique, a number of sufficient conditions, either delay independent or delay dependent, have been presented to ensure the global asymptotic stability for delayed neural networks; one can refer to [1]–[4], [6]–[13], [15]–[21], and references therein.

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Delay-dependent stability conditions, which contain information concerning time delays, are usually less conservative than delay-independent ones, especially for a neural network with a small time delay. As a result, recently, much attention has been paid on the delay-dependent stability analysis for delayed neural networks. The main aim is to derive a maximum admissible upper bound (MAUB) of the time delay such that delayed neural networks are globally asymptotically stable. The larger the MAUB of the time delay is, there is less conservatism of a delay-dependent stability criterion. In [6]–[8], some less conservative stability conditions are obtained by employing an appropriate Lyapunov–Krasovskii functional. In [11], an augmented Lyapunov–Krasovskii functional is introduced to improve the results in [6]–[8]. However, the results in [6]–[8] and [11] are still conservative due to the fact that in the Lyapunov–Krasovskii functional employed in [6]–[8] and [11], the weighting matrices $Q > 0$ and $R > 0$ in the integral terms

$$\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)R\dot{z}(s)dsd\theta + \int_{-\tau}^{0} z^T(t+s)Qz(t+s)ds \qquad (1)$$

where $z(t) \in \mathbb{R}^n$ is the system state and $R > 0$ and $Q > 0$, are kept on the whole delay interval $[-\tau, 0]$. Consequently, this feature may lead to a conservative result. On the other hand, by introducing a fraction τ/m of the time delay τ, where $m > 0$ is an integer, the following Lyapunov–Krasovskii functional [14], [15] was employed to handle the delay-dependent stability for Hopfield neural networks with time delay:

$$\check{V}(z_t) := z^T(t)\tilde{P}z(t) + \int_{-\frac{\tau}{m}}^{0} \Upsilon^T(t+s)\tilde{Q}\Upsilon(t+s)ds + \int_{-\frac{\tau}{m}}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)\tilde{R}\dot{z}(s)dsd\theta \qquad (2)$$

where $\tilde{P} > 0, \tilde{R} > 0, \tilde{Q} > 0$ and

$$\Upsilon(w) = \left[z^T(w) \;\; z^T\left(w - \frac{1}{m}\tau\right) \;\; \ldots \;\; z^T\left(w - \frac{m-1}{m}\tau\right)\right]^T.$$

Though the obtained results in [15] are improvement over some existing ones, they may not be applied to neural networks with *time-varying* delays. One of the reasons is that the weighting matrix $\tilde{R}$ is confined to just one subinterval $[-(\tau/m), 0]$, which means that it is undefined on other subintervals. Moreover, it is worth noting that the number $N(m)$ of scalar decision variables in (2) is a quadratic function of m

$$N(m) := mn(mn+1)/2 + n(n+1). \qquad (3)$$

With m increasing, $N(m)$ is quickly enlarged, which means a larger central processing unit (CPU) time cost when testing the stability criteria.

In this brief, we will *nonuniformly* decompose the whole delay interval into multiple subintervals, and construct a new Lyapunov–Krasovskii functional by choosing different weighting matrices on different subintervals. Then, we will employ the new Lyapunov–Krasovskii functional to formulate some new delay-dependent stability criteria for a class of delayed neural networks, where both constant time delays and time-varying delays will be treated. We will also give a numerical example to illustrate that the obtained results in this brief are less conservative than some existing ones.

Throughout this brief, the notations are standard. For simplicity, the symmetric term in a symmetric matrix is denoted by $\star$, e.g.,

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}.$$

II. Problem Description

Consider the delayed neural network

$$\dot{x}(t) = -Ax(t) + W_0 g(x(t)) + W_1 g(x(t-\tau(t))) + u \qquad (4)$$

where $x = [x_1(\cdot)x_2(\cdot)\ldots x_n(\cdot)]^T \in \mathbb{R}^n$ is the neuron state vector and $u = [u_1 u_2 \ldots u_n]^T \in \mathbb{R}^n$ is a constant input vector. $g(x(\cdot)) = [g(x_1(\cdot))g(x_2(\cdot))\ldots\, g(x_n(\cdot))]^T \in \mathbb{R}^n$ is the neuron activation function satisfying for $i = 1, 2, \ldots, n$

$$0 \le \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \le \nu_i \qquad \forall \xi_1, \xi_2 \in \mathbb{R}, \xi_1 \ne \xi_2 \qquad (5)$$

where ν_i $(i = 1, 2, \ldots, n)$ are known real scalars, $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ is a constant real matrix with $a_i > 0$ $(i = 1, 2, \ldots, n)$ W_0 and W_1 are the interconnection matrices representing the weighting coefficients of the neurons, and $\tau(t)$ is the time delay of the system. In this brief, two cases of $\tau(t)$, namely, constant and time varying, will be discussed, respectively.

Suppose $x^* = [x_1^* x_2^* \cdots, x_n^*]^T$ is an equilibrium of the system (4). We shift the equilibrium to the origin by changing variables

$$z_i(t) := x_i(t) - x_i^*$$
$$f_i(z_i(t)) := g_i(z_i(t) + x_i^*) - g_i(x_i^*)$$

where $i = 1, 2, \ldots, n$. Then, the system (4) is readily transformed into

$$\dot{z}(t) = -Az(t) + W_0 f(z(t)) + W_1 f(z(t-\tau(t))) \qquad (6)$$

where

$$z(t) := [z_1(t)\; z_2(t)\; \ldots\; z_n(t)]^T$$
$$f(z(t)) := [f_1(z_1(t))\;\; f_2(z_2(t))\;\; \ldots\;\; f_n(z_n(t))]^T.$$

In addition, it is easily verified from (5) that $f_i(z_i(t))$ satisfies $f_i(0) = 0$ and $\forall\, z_i \ne 0$

$$0 \le \frac{f_i(z_i)}{z_i} \le \nu_i, \qquad i = 1, 2, \ldots, n$$

which can be written as

$$f_i(z_i(\cdot))\left[f_i(z_i(\cdot)) - \nu_i z_i(\cdot)\right] \le 0, \qquad i = 1, 2, \ldots, n. \qquad (7)$$

To end this section, we introduce an integral inequality [5], which will play an important role in deriving stability criteria.

Lemma 1 [5]: For any constant matrix $X \in \mathbb{R}^{n\times n}, X = X^T > 0$, a scalar function $h := h(t) > 0$, and a vector-valued function $\dot{x} : [-h, 0] \to \mathbb{R}^n$ such that the following integration is well defined, then

$$-h\int_{-h}^{0} \dot{x}^T(t+s)X\dot{x}(t+s)ds \le \xi^T(t)\begin{bmatrix} -X & X \\ X & -X \end{bmatrix}\xi(t) \qquad (8)$$

where $\xi(t) = [x^T(t)\;\; x^T(t-h)]^T$.

III. Stability Criteria

In this section, we introduce new Lyapunov–Krasovskii functionals to derive some new delay-dependent stability criteria for the system described by (6) and (7). Both constant time delays and time-varying delays are treated, respectively. We begin with the case of the constant time delay.

A. The Case of a Constant Time Delay: $\tau(t) \equiv \tau$

Let $m > 0$ be an integer and τ_j $(j = 0, 1, 2, \ldots, m)$ be some scalars satisfying

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m = \tau$$

then the delay interval $[-\tau, 0]$ is *nonuniformly* decomposed into m segments, that is, $[-\tau, 0] = \bigcup_{j=1}^{m}[-\tau_j, -\tau_{j-1}]$. For convenience, we denote δ_j the length of the subinterval $[-\tau_j, -\tau_{j-1}]$, i.e., $\delta_j = \tau_j - \tau_{j-1}$ $(j = 1, 2, \ldots, m)$. We introduce the following Lyapunov–Krasovskii functional to deal with the asymptotic stability for systems (6) and (7):

$$V(z_t) = V_1(z_t) + V_2(z_t) + V_3(z_t) \tag{9}$$

where

$$V_1(z_t) := z^T(t)Pz(t) + 2\sum_{i=1}^{n}\lambda_i \int_0^{z_i(t)} f_i(s)ds$$

$$V_2(z_t) := \sum_{j=1}^{m} \int_{-\tau_j}^{-\tau_{j-1}} \begin{bmatrix} z(t+s) \\ f(z(t+s)) \end{bmatrix}^T \mathcal{Q}_j \begin{bmatrix} z(t+s) \\ f(z(t+s)) \end{bmatrix} ds$$

$$V_3(z_t) := \sum_{j=1}^{m} \delta_j \int_{-\tau_j}^{-\tau_{j-1}} \int_{t+\theta}^{t} \dot{z}^T(s)R_j\dot{z}(s)dsd\theta$$

with

$$P > 0 \quad R_j > 0 \quad \mathcal{Q}_j := \begin{bmatrix} Q_j & X_j \\ \star & S_j \end{bmatrix} > 0, \qquad j = 1, 2, \ldots, m$$

being real matrices of appropriate dimensions and $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$.

Remark 1: Notice that (9) is different from existing ones. The first term $V_1(z_t)$ is motivated by the Lyapunov–Krasovskii functional employed in [8] and [9]. The second and third terms $V_2(z_t)$ and $V_3(z_t)$ are constructed by using such an idea that the whole delay interval $[-\tau, 0]$ is *nonuniformly* decomposed into multiple subintervals; then, on each subinterval, we choose different weighting matrices. Another different point of (9) is that in $V_2(z_t)$ two vectors $z(\cdot)$ and $f(z(\cdot))$ are coupled by $\mathcal{Q}_j$. If we set $m = 1$ and $X_1 = 0$, then $V_2(z_t)$ reduces to the one in [8] and [9] in the case of a constant time delay. On the other hand, the number of scalar decision variables in (9) is $mn(5n+3)/2 + n(n+3)/2$, which is a *linear* function of m, not a quadratic one as the one in (3).

In this subsection, we employ (9) and the integral inequality (8) to study the stability of the system described by (6) and (7). In doing so, for simplicity, we introduce two vectors

$$\Upsilon_1 := \begin{bmatrix} z(t) \\ z(t-\tau_1) \\ \vdots \\ z(t-\tau_m) \end{bmatrix} \quad \Upsilon_2 := \begin{bmatrix} f(z(t)) \\ f(z(t-\tau_1)) \\ \vdots \\ f(z(t-\tau_m)) \end{bmatrix}. \tag{10}$$

Then, rewrite system (6) as

$$\dot{z}(t) = [\Gamma_1 \;\; \Gamma_2] \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} \tag{11}$$

where

$$\begin{cases} \Gamma_1 := [-A \;\; 0 \;\; \ldots \;\; 0] \\ \Gamma_2 := [W_0 \;\; 0 \;\; \ldots \;\; 0 \;\; W_1]. \end{cases} \tag{12}$$

Now we state and establish the following result.

Proposition 1: For a given $\tau > 0$, the origin of a system described by (6) and (7) is globally asymptotically stable if there exist

$$P > 0 \quad R_i > 0 \quad \mathcal{Q}_i := \begin{bmatrix} Q_i & X_i \\ \star & S_i \end{bmatrix} > 0, \qquad i = 1, 2, \ldots, m$$

of appropriate dimensions, and $T_j = \mathrm{diag}\{t_{j1}, t_{j2}, \ldots, t_{jn}\} \geq 0$ $(j = 0, 1, 2, \ldots, m)$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geq 0$ such that

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \delta_1\Gamma_1^T R_1 & \cdots & \delta_m\Gamma_1^T R_m \\ \star & \Phi_{22} & \delta_1\Gamma_2^T R_1 & \cdots & \delta_m\Gamma_2^T R_m \\ \star & \star & -R_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & -R_m \end{bmatrix} < 0 \tag{13}$$

where Γ_1 and Γ_2 are defined in (12), respectively, and

$$\Phi_{11} := \begin{bmatrix} \varphi_1 & R_1 & 0 & \cdots & 0 & 0 \\ \star & \varphi_2 & R_2 & \cdots & 0 & 0 \\ \star & \star & \varphi_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \star & \cdots & \varphi_m & R_m \\ \star & \star & \star & \cdots & \star & \varphi_{m+1} \end{bmatrix}$$

$$\Phi_{12} := \begin{bmatrix} \beta_1 & 0 & \cdots & 0 & PW_1 \\ 0 & \beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_m & 0 \\ 0 & 0 & \cdots & 0 & \beta_{m+1} \end{bmatrix}$$

$$\Phi_{22} := \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 & \Lambda W_1 \\ \star & \alpha_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \cdots & \alpha_m & 0 \\ \star & \star & \cdots & \star & \alpha_{m+1} \end{bmatrix}$$

with

$$\varphi_j := \begin{cases} -PA - A^TP + Q_1 - R_1, & j = 1 \\ Q_j - Q_{j-1} - R_j - R_{j-1}, & j = 2, \ldots, m \\ -Q_m - R_m, & j = m+1 \end{cases}$$

$$\alpha_j := \begin{cases} \Lambda W_0 + W_0^T\Lambda + S_1 - 2T_0, & j = 1 \\ S_j - S_{j-1} - 2T_{j-1}, & j = 2, \ldots, m \\ -S_m - 2T_m, & j = m+1 \end{cases}$$

$$\beta_j := \begin{cases} X_1 + \Pi T_0 + PW_0 - A^T\Lambda, & j = 1 \\ \Pi T_{j-1} + X_j - X_{j-1}, & j = 2, \ldots, m \\ \Pi T_m - X_m, & j = m+1 \end{cases}$$

$$\Pi := \mathrm{diag}\{\nu_1, \nu_2, \ldots, \nu_n\}.$$

Proof: Taking the time derivative of $V(z_t)$ in (9) along the trajectory of (11) yields

$$\dot{V}(z_t) = \dot{V}_1(z_t) + \dot{V}_2(z_t) + \dot{V}_3(z_t) \tag{14}$$

where

$$\dot{V}_1(z_t) = z^T(t)(-PA - A^TP)z(t) + 2z^T(t)(PW_0 - A^T\Lambda) \times f(z(t)) + 2z^T(t)PW_1 f(z(t-\tau)) + f^T(z(t))\left(\Lambda W_0 + W_0^T\Lambda\right) f(z(t)) + 2f^T(z(t))\Lambda W_1 f(z(t-\tau))$$

$$\dot{V}_2(z_t) = \sum_{j=1}^{m} \begin{bmatrix} z(t-\tau_{j-1}) \\ f(z(t-\tau_{j-1})) \end{bmatrix}^T \mathcal{Q}_j \begin{bmatrix} z(t-\tau_{j-1}) \\ f(z(t-\tau_{j-1})) \end{bmatrix} - \sum_{j=1}^{m} \begin{bmatrix} z(t-\tau_j) \\ f(z(t-\tau_j)) \end{bmatrix}^T \mathcal{Q}_j \begin{bmatrix} z(t-\tau_j) \\ f(z(t-\tau_j)) \end{bmatrix}$$

$$\dot{V}_3(z_t) = \dot{z}^T(t)\sum_{j=1}^{m}\delta_j^2 R_j \dot{z}(t) - \sum_{j=1}^{m}\delta_j \int_{t-\tau_j}^{t-\tau_{j-1}} \dot{z}^T(s)R_j\dot{z}(s)ds.$$

Apply Lemma 1 to obtain

$$-\delta_j \int_{t-\tau_j}^{t-\tau_{j-1}} \dot{z}^T(s)R_j\dot{z}(s)ds \le \begin{bmatrix} z(t-\tau_{j-1}) \\ z(t-\tau_j) \end{bmatrix}^T \begin{bmatrix} -R_j & R_j \\ R_j & -R_j \end{bmatrix} \begin{bmatrix} z(t-\tau_{j-1}) \\ z(t-\tau_j) \end{bmatrix}. \quad (15)$$

Notice that from (7), for any $t_{ji} \ge 0$ $(j = 0,1,2,\ldots,m)$ the following is true for any $i \in \{1,2,\ldots,n\}$:

$$t_{ji} f_i(z_i(t-\tau_j))\left[f_i(z_i(t-\tau_j)) - \nu_i z_i(t-\tau_j)\right] \le 0$$

which is rewritten as

$$0 \le 2z^T(t-\tau_j)\Pi T_j f(z(t-\tau_j)) - 2f^T(z(t-\tau_j))T_j f(z(t-\tau_j)). \quad (16)$$

From (14)–(16), after simple algebraic manipulation, we have

$$\dot{V}(t,z_t) \le \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}^T \Phi \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} \quad (17)$$

where

$$\Phi := \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} + \begin{bmatrix} \Gamma_1^T \\ \Gamma_2^T \end{bmatrix} \sum_{j=1}^{m}\delta_j^2 R_j \begin{bmatrix} \Gamma_1^T \\ \Gamma_2^T \end{bmatrix}^T.$$

Notice that if (13) is feasible, then $\Phi < 0$ by using Schur complement. Let $\varepsilon = \lambda_{\min}(-\Phi)$; it is clear from (17) that $\dot{V}(z_t) \le -\varepsilon z^T(t)z(t) < 0$ for $z(t) \neq 0$, from which we can conclude that the system described by (6) and (7) is globally asymptotically stable, which completes the proof. □

B. *The Case of a Time-Varying Delay:* $0 < \tau(t) \le \tau < \infty$

For simplicity of the presentation, we introduce two vectors as

$$\tilde{\Upsilon}_1 := \begin{bmatrix} \Upsilon_1 \\ z(t-\tau(t)) \end{bmatrix} \quad \tilde{\Upsilon}_2 := \begin{bmatrix} \Upsilon_2 \\ f(z(t-\tau(t))) \end{bmatrix} \quad (18)$$

where Υ_1 and Υ_2 are defined in (10), and rewrite the system (6) as

$$\dot{z}(t) = [\tilde{\Gamma}_1\ \tilde{\Gamma}_2]\begin{bmatrix} \tilde{\Upsilon}_1 \\ \tilde{\Upsilon}_2 \end{bmatrix} \quad (19)$$

where

$$\begin{cases} \tilde{\Gamma}_1 := [-A\ 0\ \ldots\ 0] \\ \tilde{\Gamma}_2 := [W_0\ 0\ \ldots\ 0\ W_1]. \end{cases} \quad (20)$$

In this section, $\tau(t)$ is treated as the following two subcases.

Subcase 1: $\tau(t)$ is a continuous function satisfying

$$0 < \tau(t) \le \tau < \infty \qquad \forall t \ge 0. \quad (21)$$

Subcase 2: $\tau(t)$ is a differentiable function satisfying

$$0 < \tau(t) \le \tau < \infty \quad \dot{\tau}(t) \le \mu < \infty \qquad \forall t \ge 0 \quad (22)$$

where τ and μ are two scalars. In subcase 1, employing the Lyapunov–Krasovskii functional (9) again, we establish the following.

Proposition 2: Under subcase 1, for a given scalar $\tau > 0$, the origin of system described by (6) and (7) is globally asymptotically stable if there exist matrices

$$P > 0 \quad R_i > 0 \quad \mathcal{Q}_i := \begin{bmatrix} Q_i & X_i \\ \star & S_i \end{bmatrix} > 0, \qquad i = 1,2,\ldots,m$$

of appropriate dimensions, and $T_j = \mathrm{diag}\{t_{j1}, t_{j2}, \ldots, t_{jn}\} \ge 0$ $(j = 0,1,2,\ldots,m)$ and $\Lambda = \mathrm{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_n\} \ge 0$, $L = \mathrm{diag}\{l_1,l_2,\ldots,l_n\} \ge 0$ such that for $\forall\, k \in \{1,2,\ldots,m\}$

$$\begin{bmatrix} \Omega_{11}+\Omega_{11}^k & \Omega_{12} & \delta_1\tilde{\Gamma}_1^T R_1 & \cdots & \delta_m\tilde{\Gamma}_1^T R_m \\ \star & \Omega_{22} & \delta_1\tilde{\Gamma}_2^T R_1 & \cdots & \delta_m\tilde{\Gamma}_2^T R_m \\ \star & \star & -R_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & -R_m \end{bmatrix} < 0 \quad (23)$$

where

$$\Omega_{11} := \begin{bmatrix} \varphi_1 & R_1 & 0 & \cdots & 0 & 0 & 0 \\ \star & \varphi_2 & R_2 & \cdots & 0 & 0 & 0 \\ \star & \star & \varphi_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \star & \star & \star & \cdots & \varphi_m & R_m & 0 \\ \star & \star & \star & \cdots & \star & \varphi_{m+1} & 0 \\ \star & \star & \star & \cdots & \star & \star & 0 \end{bmatrix}$$

$$\Omega_{12} := \begin{bmatrix} \beta_1 & 0 & \cdots & 0 & PW_1 \\ 0 & \beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{m+1} & 0 \\ 0 & 0 & \cdots & 0 & \Pi L \end{bmatrix}$$

$$\Omega_{22} := \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 & \Lambda W_1 \\ \star & \alpha_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \cdots & \alpha_{m+1} & 0 \\ \star & \star & \cdots & \star & -2L \end{bmatrix}$$

$$\Omega_{11}^k := \left(\psi_{ij}^k\right)_{(m+2)\times(m+2)}$$

$$\psi_{ij}^k := \begin{cases} -R_k, & i = k, \quad j = k+1 \\ R_k, & i \in \{k,k+1\}, \quad j = m+2 \\ -2R_k, & i = m+2, \quad j = m+2 \\ 0, & \text{otherwise} \end{cases}$$

with $\varphi_j, \beta_j, \alpha_j, (j = 1,2,\ldots,m+1)$ being defined in Proposition 1.

Proof: Taking the time derivative of $V(z_t)$ in (9) along the trajectory of (19), we have

$$\begin{aligned}\dot{V}(z_t) &= z^T(t)(-PA - A^TP)z(t) + \dot{z}^T(t)\sum_{j=1}^{m}\delta_j^2 R_j \dot{z}(t)\\ &+ 2z^T(t)(PW_0 - A^T\Lambda)f(z(t))\\ &+ 2z^T(t)PW_1 f(z(t-\tau(t)))\\ &+ f^T(z(t))\left(\Lambda W_0 + W_0^T\Lambda\right)f(z(t))\\ &+ 2f^T(z(t))\Lambda W_1 f(z(t-\tau(t)))\\ &+ \sum_{j=1}^{m}\begin{bmatrix}(t-\tau_{j-1})\\ f(z(t-\tau_{j-1}))\end{bmatrix}^T \mathcal{Q}_j \begin{bmatrix}z(t-\tau_{j-1})\\ f(z(t-\tau_{j-1}))\end{bmatrix}\\ &- \sum_{j=1}^{m}\begin{bmatrix}z(t-\tau_j)\\ f(z(t-\tau_j))\end{bmatrix}^T \mathcal{Q}_j \begin{bmatrix}z(t-\tau_j)\\ f(z(t-\tau_j))\end{bmatrix}\\ &- \sum_{j=1}^{m}\delta_j \int_{t-\tau_j}^{t-\tau_{j-1}} \dot{z}^T(s)R_j\dot{z}(s)ds.\end{aligned}$$

For any $t \geq 0$, there should exist an integer $k \in \{1,2,\ldots,m\}$ such that $\tau(t) \in [\tau_{k-1}, \tau_k]$. In this situation, apply Lemma 1 to obtain

$$\begin{aligned}&-\delta_k \int_{t-\tau_k}^{t-\tau_{k-1}} \dot{z}^T(s)R_k\dot{z}(s)ds\\ &= -\delta_k \int_{t-\tau_k}^{t-\tau(t)} \dot{z}^T(s)R_k\dot{z}(s)ds - \delta_k \int_{t-\tau(t)}^{t-\tau_{k-1}} \dot{z}^T(s)R_k\dot{z}(s)ds\\ &\leq -(\tau_k - \tau(t)) \int_{t-\tau_k}^{t-\tau(t)} \dot{z}^T(s)R_k\dot{z}(s)ds - (\tau(t) - \tau_{k-1})\\ &\quad\times \int_{t-\tau(t)}^{t-\tau_{k-1}} \dot{z}^T(s)R_k\dot{z}(s)ds\\ &\leq \begin{bmatrix}z(t-\tau_{k-1})\\ z(t-\tau_k)\\ z(t-\tau(t))\end{bmatrix}^T \begin{bmatrix}-R_k & 0 & R_k\\ \star & -R_k & R_k\\ \star & \star & -2R_k\end{bmatrix}\\ &\quad\times \begin{bmatrix}z(t-\tau_{k-1})\\ z(t-\tau_k)\\ z(t-\tau(t))\end{bmatrix}.\end{aligned} \tag{24}$$

For $j \neq k$, using Lemma 1 again yields

$$\begin{aligned}-\delta_j \int_{t-\tau_j}^{t-\tau_{j-1}} \dot{z}^T(s)R_j\dot{z}(s)ds &\leq \begin{bmatrix}z(t-\tau_{j-1})\\ z(t-\tau_j)\end{bmatrix}^T\\ &\quad\times \begin{bmatrix}-R_j & R_j\\ \star & -R_j\end{bmatrix}\begin{bmatrix}z(t-\tau_{j-1})\\ z(t-\tau_j)\end{bmatrix}.\end{aligned}$$

From (7), it is easy to see that for any $t_{ji} \geq 0$ $(j = 0,1,2,\ldots,m)$ and $l_i \geq 0$, the following are true for any $i \in \{1,2,\ldots,n\}$:

$$\begin{aligned}2t_{ji}f_i(z_i(t-\tau_j))\left[f_i(z_i(t-\tau_j)) - \nu_i z_i(t-\tau_j)\right] &\leq 0\\ 2l_i f_i(z_i(t-\tau(t)))\left[f_i(z_i(t-\tau(t))) - \nu_i z_i(t-\tau(t))\right] &\leq 0\end{aligned}$$

which result in

$$\begin{aligned}0 \leq &\sum_{j=0}^{m} 2z^T(t-\tau_j)\Pi T_j f(z(t-\tau_j))\\ &- \sum_{j=0}^{m} 2f^T(z(t-\tau_j))T_j f(z(t-\tau_j))\\ &+ 2z^T(t-\tau(t))\Pi L f(z(t-\tau(t)))\\ &- 2f^T(z(t-\tau(t)))L f(z(t-\tau(t))).\end{aligned}$$

After some algebraic manipulation, we have

$$\dot{V}(z_t) \leq \begin{bmatrix}\tilde{\Upsilon}_1\\ \tilde{\Upsilon}_2\end{bmatrix}^T \Omega_k \begin{bmatrix}\tilde{\Upsilon}_1\\ \tilde{\Upsilon}_2\end{bmatrix} \tag{25}$$

where

$$\Omega_k := \begin{bmatrix}\Omega_{11} + \Omega_{11}^k & \Omega_{12}\\ \star & \Omega_{22}\end{bmatrix} + \begin{bmatrix}\tilde{\Gamma}_1^T\\ \tilde{\Gamma}_2^T\end{bmatrix}\sum_{j=1}^{m}\delta_j^2 R_j \begin{bmatrix}\tilde{\Gamma}_1^T\\ \tilde{\Gamma}_2^T\end{bmatrix}^T$$

with $\Omega_{11}, \Omega_{11}^k, \Omega_{12}, \Omega_{22}$ being defined in (23). A sufficient condition for global asymptotic stability of systems (6) and (7) is that

$$P > 0 \quad R_i > 0 \quad \mathcal{Q}_i := \begin{bmatrix}Q_i & X_i\\ \star & S_i\end{bmatrix} > 0, \qquad i = 1,2,\ldots,m$$

of appropriate dimensions, and $T_j = \mathrm{diag}\{t_{j1}, t_{j2}, \ldots, t_{jn}\} \geq 0$ $(j = 0,1,2,\ldots,m)$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geq 0$, $L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\} \geq 0$ such that

$$\begin{aligned}\dot{V}(z_t) &\leq \begin{bmatrix}\tilde{\Upsilon}_1\\ \tilde{\Upsilon}_2\end{bmatrix}^T \Omega_k \begin{bmatrix}\tilde{\Upsilon}_1\\ \tilde{\Upsilon}_2\end{bmatrix}\\ &\leq -\lambda z^T(t)z(t) < 0 \qquad \forall z(t) \neq 0\end{aligned} \tag{26}$$

where $\lambda > 0$. In order to guarantee (26), we require the condition that $\Omega_k < 0$, which is equivalent to

$$\begin{bmatrix}\Omega_{11} + \Omega_{11}^k & \Omega_{12} & \delta_1\tilde{\Gamma}_1^T R_1 & \cdots & \delta_m\tilde{\Gamma}_1^T R_m\\ \star & \Omega_{22} & \delta_1\tilde{\Gamma}_2^T R_1 & \cdots & \delta_m\tilde{\Gamma}_2^T R_m\\ \star & \star & -R_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \star & \star & \star & \cdots & -R_m\end{bmatrix} < 0.$$

Considering all the possibilities of k in the set $\{1,2,\ldots,m\}$, we arrive at the condition that (23) holds for any $k \in \{1,2,\ldots,m\}$, which completes the proof. □

From the above proof of Proposition 2, it is clear that the Lyapunov–Krasovskii functional (9) can be used to handle the case when the time delay is time varying. One main reason is that the relationship between the states $z(t), z(t-\tau_1), \ldots, z(t-\tau_m)$ and the state $z(t-\tau(t))$ are well established by (24), which is due to the term $V_3(z_t)$ in (9). If we replace $V_3(z_t)$ with $\delta_1 \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)R_1\dot{z}(s)dsd\theta$, which is similar to the first term in (2), then the relationship in (24) cannot be built because the time-varying delay $\tau(t)$ is not always on the subinterval $[-\tau_1, 0]$.

In subcase 2, modify (9) as

$$\tilde{V}(z_t) = V(z_t) + V_4(z_t) \tag{27}$$

where $V(z_t)$ is defined in (9) and

$$V_4(z_t) := \int_{t-\tau(t)}^{t} \begin{bmatrix}z(s)\\ f(z(s))\end{bmatrix}^T \begin{bmatrix}Q_0 & X_0\\ \star & S_0\end{bmatrix}\begin{bmatrix}z(s)\\ f(z(s))\end{bmatrix}ds$$

with

$$\begin{bmatrix} Q_0 & X_0 \\ \star & S_0 \end{bmatrix} > 0$$

to be determined. Similar to the proof of Proposition 2, we have the following.

Proposition 3: Under subcase 2, for given scalars $\tau > 0$ and μ, the origin of system described by (6) and (7) is globally asymptotically stable if there exist matrices

$$P > 0 \quad R_i > 0 \quad \mathcal{Q}_j := \begin{bmatrix} Q_j & X_j \\ \star & S_j \end{bmatrix} > 0, \qquad i = 1, 2, \ldots, m$$

of appropriate dimensions, and $T_j = \mathrm{diag}\{t_{j1}, t_{j2}, \ldots, t_{jn}\} \geq 0$ $(j = 0, 1, 2, \ldots, m)$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geq 0$, $L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\} \geq 0$ such that for $\forall\, k \in \{1, 2, \ldots, m\}$

$$\begin{bmatrix} \tilde{\Omega}_{11} + \Omega_{11}^k & \tilde{\Omega}_{12} & \delta_1 \tilde{\Gamma}_1^T R_1 & \cdots & \delta_m \tilde{\Gamma}_1^T R_m \\ \star & \tilde{\Omega}_{22} & \delta_1 \tilde{\Gamma}_2^T R_1 & \cdots & \delta_m \tilde{\Gamma}_2^T R_m \\ \star & \star & -R_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & -R_m \end{bmatrix} < 0$$

where

$$\tilde{\Omega}_{11} := \Omega_{11} + \mathrm{diag}\left\{Q_0, 0, \ldots, 0, -(1-\mu)Q_0\right\}$$
$$\tilde{\Omega}_{12} := \Omega_{12} + \mathrm{diag}\left\{X_0, 0, \ldots, 0, -(1-\mu)X_0\right\}$$
$$\tilde{\Omega}_{22} := \Omega_{22} + \mathrm{diag}\left\{S_0, 0, \ldots, 0, -(1-\mu)S_0\right\}$$

with Ω_{11}^k, $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, Ω_{11}, Ω_{12}, and Ω_{22} being defined in Proposition 2.

Remark 2: When the time delay is constant, by employing a Lyapunov–Krasovskii functional [14], [15], some less conservative delay-dependent criteria were obtained in [15], but the results in [15] are applicable just to delayed Hopfield neural networks, not to system (4) with $W_0 \neq 0$. Furthermore, when the time delay is in subcase 2, the authors claimed in [15, Remark 3] that just modifying the Lyapunov–Krasovskii functional by replacing the constant delay τ with the time-varying delay $\tau(t)$ can yield LMI-based delay-dependent conditions without introducing any other assumptions. Unfortunately, this claim is not true. In fact, by replacing τ with $\tau(t)$, the corresponding Lyapunov–Krasovskii functional can be written as

$$\bar{V}(z_t) := z^T(t)Pz(t) + \int_{-\frac{\tau(t)}{m}}^{0} \int_{t+\theta}^{t} \dot{z}^T(s)\tilde{R}\dot{z}(s)\,ds\,d\theta + \int_{t-\frac{\tau(t)}{m}}^{t} \tilde{\Upsilon}^T(s)\tilde{Q}\tilde{\Upsilon}(s)\,ds$$

where

$$\tilde{\Upsilon}(w) = \begin{bmatrix} z(w) \\ z\left(w - \frac{1}{m}\tau(w)\right) \\ \vdots \\ z\left(w - \frac{m-1}{m}\tau(w)\right) \end{bmatrix}.$$

TABLE I
ACHIEVED MAUBs FOR A CONSTANT TIME DELAY

Method	MAUB
He, Liu and Rees [8]	3.5841
He, Liu, Rees and Wu [9]	3.5841
Li, Guo, Sun and Lin [11]	3.5898
Proposition 1 $(m = 2)$	4.1880
Proposition 1 $(m = 3)$	4.2843
Proposition 1 $(m = 5)$	4.3340
Proposition 1 $(m = 10)$	4.3531

Now, taking the derivative of $\bar{V}(z_t)$ on t, we have the corresponding derivative of the third term as

$$\tilde{\Upsilon}^T(t)\tilde{Q}\tilde{\Upsilon}(t) - \left(1 - \frac{\dot{\tau}(t)}{m}\right)\tilde{\Upsilon}^T\left(t - \frac{\tau(t)}{m}\right)\tilde{Q}\tilde{\Upsilon}\left(t - \frac{\tau(t)}{m}\right)$$

which results in some new delay states shown in the following:

$$\tilde{\Upsilon}\left(t - \frac{\tau(t)}{m}\right) = \begin{bmatrix} z\left(t - \frac{\tau(t)}{m}\right) \\ z\left(t - \frac{\tau(t)}{m} - \frac{1}{m}\tau\left(t - \frac{\tau(t)}{m}\right)\right) \\ \vdots \\ z\left(t - \frac{\tau(t)}{m} - \frac{m-1}{m}\tau\left(t - \frac{\tau(t)}{m}\right)\right) \end{bmatrix}.$$

These new delay states $z(t - (\tau(t)/m) - (j/m)\tau(t - (\tau(t)/m)))$ $(j = 1, 2, \ldots, m-1)$ will make the delay-dependent stability analysis more complicated, which needs to be further investigated. Therefore, without any other assumptions, the claim in Lemma 3 in [15] is not realistic.

Remark 3: Propositions 1–3 provide some new delay-dependent stability criteria for delayed neural networks based on the delay decomposition approach. With m increasing, the dimensions of the involved LMIs are fast growing, which will lead to more CPU computing time. However, it can be easily seen from Section IV that the larger m is, there will be less conservatism of the results. As a compromise, for practical neural networks, we can take $m = 2$ or $m = 3$ to obtain some results, which are still less conservative than those in some existing literature.

IV. A NUMERICAL EXAMPLE

In this section, a numerical example is given to illustrate the effectiveness of the proposed results. Consider the delayed neural network (4) with

$$A = \mathrm{diag}\{1.2769,\ 0.6231,\ 0.9230,\ 0.4480\}$$
$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}$$
$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & 0.2775 \end{bmatrix}$$
$$\nu_1 = 0.1137 \quad \nu_2 = 0.1279 \quad \nu_3 = 0.7994 \quad \nu_4 = 0.2368.$$

Suppose the time delay is a constant time delay. Using the criterion in [16], no conclusion can be made. Employing criteria in [8], [9], and [11], the maximum admissible upper bounds (MAUBs), $\tau_{\max}$, are listed in Table I. Applying Proposition 1 with $\delta_1 = \delta_2 = \cdots = \delta_m =$

TABLE II
ACHIEVED MAUBS FOR A TIME-VARYING DELAY

Method	$\mu = 0.1$	$\mu = 0.3$
He, Liu, Rees and Wu [9]	3.3039	2.8810
Li, Guo, Sun and Lin [11]	3.2819	2.7326
Proposition 3 $(m = 2)$	3.5423	2.9677
Proposition 3 $(m = 3)$	3.5709	3.0057
Proposition 3 $(m = 5)$	3.5989	3.0310

(τ/m) yields some results for different values of m, which are also listed in the table. From this table, one can see that Proposition 1 provides larger MAUBs than the those criteria in [8], [9], and [11] and the larger m is, the larger $\tau_{\max}$ becomes. We also computed the cost of CPU time for $m = 3$ and $m = 10$; the results are 84 and 2844 s, respectively. Apparently, the difference between the values of $\tau_{\max}$ for $m = 10$ and $m = 3$ is just 0.0688, but the CPU time cost by the former is almost 34 times larger than the cost by the latter. As a compromise, taking $m = 3$ is a good choice for the obtained MAUB and for the cost of the CPU time.

Next, let us assume that the time delay $\tau(t)$ is in subcase 2. For $\mu = 0.1$ and $\mu = 0.3$, the obtained MAUBs by applying the criteria in [9] and [11] and Proposition 3 with $\delta_1 = \delta_2 = \cdots = \delta_m = (\tau/m)$ are listed in Table II, from which it is clear that this paper yields much less conservative results than those in [9] and [11], which demonstrates the effectiveness of the criterion in this paper. It should be mentioned that no conclusion can be made by using the criteria in [3] and [4].

V. CONCLUSION

The problem of global asymptotic stability for delayed neural networks has been addressed by using new Lyapunov–Krasovskii functional, which has been constructed by nonuniformly dividing the whole delay interval into multiple segments. Some new delay-dependent stability criteria have been derived for neural networks with both constant time delays and time-varying delays. A numerical example has shown that these new stability criteria are less conservative than some existing ones in the literature.

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