Abstract

The aim of this research is to develop a mathematical framework for the derivation of the transition matrix between symmetric group bases. This problem arises in the study of quantum mechanics. To do this, we follow the current methods of abstract algebra, specifically group theory and group representation theory.

We assume a knowledge of basic group theory and of group representation theory, including matrix representation theory, modular representation theory and group character theory. We introduce the basic concepts of the representation theory of the symmetric group. This includes tableaux, Young tableaux, Young subgroups, tabloids and polytabloids.

We then discuss combinatorial algorithms on tableaux. We describe symmetric functions and their relation to the representation theory of the symmetric group. We introduce the Young-Yamanouchi representation and its adaptation to the representation of partial skew tableaux. A survey of current research into the transition matrix between symmetric group bases is then given.

We develop the research by Hamel et al. [7] and McAven et al. [8]. In this work the decomposition of a tableau into component tableaux is examined. The justication of the component tableaux into normal tableaux is given. The approach is to apply set-theoretic and combinatorial concepts to the decomposition of a tableau. This establishes a formal and rigorous mathematical frame-work in which further research may be undertaken.

In this work we have considered the decomposition of a partial skew tableau into a tuple of partial skew tableaux. We have also considered the decomposition of a partial

skew tableau into partial normal tableaux. The decomposition of a Young tableau may be considered to be special cases of these decompositions. Thus the decomposition given by Hamel et al. [7] and McAven et al. [8] is a special case of our more general mathematical framework. We have developed the set-theoretic and combinatorial aspects of these decompositions, in order to facilitate further research in this area.

Preface

In this thesis, we attempt to find the transition matrix between symmetric group bases. We begin with a survey of background material in this area.

We assume a knowledge of group representation theory, including group theory, matrix representation theory, modular representation theory, group character theory and the group algebra.

We then discuss tableaux, Young tableaux, Young subgroups, tabloids and polytabloids. We give an introduction to combinatorial algorithms in group representation theory. We introduce symmetric functions and the Young-Yamanouchi representation of the symmetric group. We outline the skew representation, which is an adaptation of the Young-Yamanouchi representation for partial skew tableaux. We give a summary of two papers by Hamel et al. [7] and McAven et al. [8], which investigate the transition matrix. Then the author begins his own research into the transition matrix.

In the thesis, we present a number of theorems from the literature. These theorems are presented without proof for two reasons. First, we wish to limit the length of the thesis by omitting proofs. Second, we wish to avoid cluttering the thesis with unimportant theorems which serve only to aid in proving the main theorems. The proofs of most of the theorems given here invoke many other theorems, which in turn invoke other theorems in their proofs, and so on. These other theorems are not important in themselves, but serve only to prove the main results. By omitting proofs, we can omit unimportant theorems and concentrate on stating important results.

Towards Determining the Transition Matrix between Symmetric Group Bases

by

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Declaration

This thesis contains no material that has been accepted for the award of another degree. Furthermore, to the best of my knowledge, this thesis does not contain any material previously published or written by another person, except where due reference is made in the text.

.....Ross Mortensen.........

Ross Mortensen

Glossary

Symbol	Meaning	Page
α	Ordered decomposition of a partial skew tableau	149
c	Rotation of vertices	3
$\mathbb{C}[[\boldsymbol{x}]]$	The ring of formal power series of $\boldsymbol{x}$ over $\mathbb{C}$	50
ch^n	The characteristic map	57
$(c_1, c_2, \cdots, c_l)$	The slide sequence for a tableau	46
C_t	Column stabilizer group of Young tableau	23
$c^{\lambda}_{\mu\nu}$	Littewood Richardson coefficient	63
e	Identity rotation	2
e_n	n^{th} Elementary symmetric function	52
e_t	n^{th} Polytabloid associated with Young tableau	24
$\approxeq$	Relation of tabloid having same skew shape	158
$\approxeq$	Relation of tuples of tableau having pairwise same skew shape	163
f^{λ}	Number of standard λ-tableau	34
f_{λ}	Number of standard skew tableaux	87
$\lambda \trianglerighteq \mu$	Dominance order of partitions	55
$h_{i,j}$	Hooklength of cell (i, j)	33
$H_{i,j}$	Hook of cell (i, j)	33
h_n	n^{th} complete homogeneous symmetric function	52
$j_c(t)$	Forward slide of tableau t into cell c	44
$j^c(t)$	Backward slide of tableau t into cell c	45

Glossary

Symbol	Meaning	Page
$\overset{s}{\cong}$	s– equivalence of tableaux	41
$\overset{1}{\cong}$	Knuth relation of the first kind	42
$\overset{2}{\cong}$	Knuth relation of the second kind	42
$\overset{K}{\cong}$	Knuth equivalence of permutations	43
$\overset{K}{\cong}$	Knuth equivalence of tableaux	48
κ_t	Signed column sum for Youngs tableaux t	24
$K_{\lambda\mu}$	Kosta number	30
λ	Integer partition	16
λ/μ	Skew shape	41
$\lambda_1 \times \lambda_2 \times \cdots \times \lambda_h$	Direct product of component tableaux	87
$[\lambda_1] \otimes [\lambda_2] \otimes \cdots \otimes [\lambda_h]$	Tensor product of representations associated with component tableaux	89
$<$	Dictionary order of tableaux	70
$<$	Partition order of tableaux	71
M^λ	Permutation module associated with λ	19
μ	Composition	22
$[\mu].[\nu]$	Outer product of irreducible representations	101
m_λ	Monomial symmetric function	50
$M^\lambda_{(k-1,k)}$	Young-Yamamouchi representation matrix for transposition $(k-1,k)$	80

Glossary

Symbol	Meaning	Page
$\overline{M}^{\lambda}_{(k-1,k)}$	Dual Young-Yamamouchi representation matrix for $(k-1,k)$	111
$N \times N \times \ldots \times N$	Set of tuples of normal tableaux	172
$\pi \overset{R-S}{\leftrightarrow} (s,t)$	Robinson-Schensted map	35
π_t	Row word of tableau t	39
p_n	n^{th} power symmetric function	52
P	Permutation matrix	115
r	Rotation of vertices	1
r_x	Row insertion operator for integer x	37
R_t	Row stabilizer group of Youngs tableau t	23
R_{ik}	Youngs raising operator	89
$\wedge$	Ring of symmetric functions	51
$R(G)$	Set of class functions on group G	57
R^n	Space of class functions on S_n	57
ρ	Reciprocal of axial distance	79
$s(\pi)$	Insertion tableau for permutation π	38
$\overset{s}{\cong}$	s-equivalence of tableaux	41
S^{λ}	Specht module	25
S_{λ}	Young subgroup associated with partition λ	17
$sh\, t$	Shape of Young tableau t	18

Glossary

Symbol	Meaning	Page
$s_\lambda(\mathbf{x})$	Schur function of partition λ	54
$< s_\lambda, s_\mu >$	Inner product on $\wedge^n$	57
$t(\pi)$	Recording tableau on permutation π	39
t	Young tableau	17
t	Tabloid	18
T	Generalised Young tabaleau	29
T	Transformation matrix	115
T	Set of partial skew tableau	149
$T_{\lambda\mu}$	Generalised tabaleau of shape λ	29
T_n	Set of partial skew tableau having n nodes	149
$T_{\lambda/\upsilon}$	Set of partial skew tableau having shape λ/υ	149
$\tau^0_{\lambda\mu}$	Semi-standard tableau of shape λ	30
τ_{ij}	Axial distance between nodes in a tableau	79
$T \times T \times \cdots \times T$	Set of tuples of partial skew tableaux	149
$\mathbf{x}^\mu$	Monomial weight in $\mathbb{C}[[\boldsymbol{x}]]$	54
$\mathbf{x}^T$	Weight of generalised tableau T	54
U	Set of positive integers	148
V^λ	Vector space	6
z_μ	Centraliser of a permutation	57

Chapter 1

Introduction

Group theory arises in the study of physical systems, such as quantum physics and crystallography. Group theory provides a mathematical tool for the study of symmetry.

We begin with an example of a physical structure, as shown in Figure 1.1, known as the trigonal bipyramid. The vertices are numbered from 1 to 5. We can move this solid such that the geometric appearance is retained, but the vertices are interchanged. Let r denote the rotation through 120° about a vertical axis through vertices 1 and 5. Vertices 1 and 5 remain unchanged. Vertex 2 goes to the position formerly occupied by vertex 3, 3 to that of 4 and 4 goes to the former position of 2.

This rotation applied twice is written as r^2 and this rotates the solid through 240° The rotation applied three times (r^3) is a rotation through 360°, which is indistinguishable

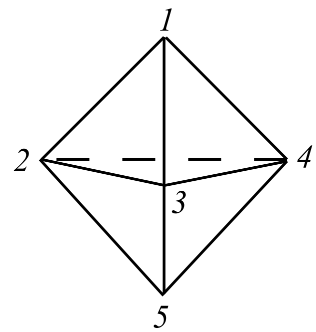


Figure 1.1: Trigonal Pyramid

from no rotation at all. This acts as an identity operation, so we may write

$$r^3 = e$$

where e is the identity operation.

We may write the rotation r as

$$r = (2\ 3\ 4)$$

which means that vertex 2 goes to vertex 3, 3 to 4, and 4 to 2. We omit vertices 1 and 5 as they are left unchanged. Similarly we may write

$$r^2 = (2\ 4\ 3)$$

and we call such a motion a cycle.

There is also a motion which interchanges vertices 1 and 5. Consider a rotation through 180° about an axis through vertex 4 and the midpoint of the edge joining vertices 2 and

3. The geometrical appearance of the solid is retained. As vertices 2 and 3 are also interchanged, this rotation is written as

$$c = (1\ 5)(2\ 3).$$

We note that c^2 is a rotation through 360°, so that

$$c^2 = e.$$

We say that c is the product of two cycles $(1\ 5)$ and $(2\ 3)$.

Two more rotation are possible: one about an axis through vertex 2 and one about an axis through vertex 3. We note that

$$rc = (1\ 5)(3\ 4)$$

so that rc is a rotation through 180° about an axis through vertex 2. Similarly

$$r^2c = (1\ 5)(2\ 4)$$

so that r^2c is a 180° rotation about an axis through vertex 3.

This leads to the following group table

We say that the set e, c, rc, r^2, r, r^2c together with the operation of successive motions forms a group. This group has the order 6.

This simple example provides an illustration of how group theory provides a mathematical

Table 1.1:

	e	c	rc	r^2	r	r^2c
e	e	c	rc	r^2	r	r^2c
c	c	e	r^2	rc	r^2 c	r
rc	rc	r	e	r^2 c	c	r^2
r^2	r^2	r^2 c	c	r	e	rc
r	r	rc	r^2 c	e	r^2	c
r^2 c	r^2 c	r^2	r	c	rc	e

tool for the study of the symmetry of solids in crystallography. Symmetry also plays a part in the structure of molecules.

We may use permutation groups to describe symmetries of molecular structures. We may label elements of a molecular structure by the integers $1, 2, \ldots . n$. Then we may describe the symmetries of the molecular structure by a permutation of the labels in cycle notation, for example

$$(2\,3 \ldots n\,1)$$

would translate element 2 to 3, 3 to 4 and so on. In quantum physics, we would like to use this permutation notation to describe symmetries among angular momentum states.

The symmetric group, S_n, on n elements, arises in the study of the quantum physics of the many-electron atom. In pursuit of this study, group representations play an important role.

The number of irreducible representations of a symmetric group is equal to the number of conjugacy classes of the symmetric group. This is the number of partitions of n. A partition is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell) \quad \text{such that} \quad \sum_i \lambda_i = n \quad \text{and} \quad \lambda_i \geq \lambda_j \text{ for } i < j.$$

The subgroup S_λ of S_n is isomorphic to the direct product

$$S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}.$$

A set of representations can be produced by inducing the trivial representation on each S_λ up to S_n. The symmetric group, S_n, is used in quantum mechanics to compare systems differing in the arrangement of the occupants of the n states. For this reason, basis vectors of representations of S_n are studied. A system may have symmetries with respect to the first a particles and the last b particles. Hence it becomes necessary to consider basis function of the direct product subgroup $S_a \times S_b$, $a+b=n$. When $b=1$, we have the Young-Yamanouchi basis which is standard. When $b \neq 1$, the basis is non-standard.

The representation matrices for such a basis can be written as a matrix product of the representation matrices in the standard basis and a transformation matrix, or transition matrix, which transforms elements of the standard basis to elements of the $S_a \times S_b$ basis. Since the symmetric group is generated by adjacent transpositions, it suffices to consider the transformation coefficients for these transpositions.

The Young-Yamanouchi basis is standard and exists in the literature [12]. Its derivation works recursively by assuming the irreducible representations are known for the subgroup

S_{n-1}. Thus it corresponds to the subgroup chain

$$S_{n-1} \times S_1 \supset S_{n-2} \times S_1 \times S_1 \supset \cdots \supset S_1 \times S_1 \times \cdots S_1,$$

obtained by successive removal of the S_1 subgroups.

We consider the Young subgroup and the induced representation. Let $\pi_1, \pi_2, \cdots, \pi_k$ be a transversal for the Young subgroup S_λ. The vector space

$$V^\lambda = C[\boldsymbol{\pi}_1 \boldsymbol{S}_\lambda, \boldsymbol{\pi}_2 \boldsymbol{S}_\lambda, \cdots, \boldsymbol{\pi}_k \boldsymbol{S}_\lambda]$$

where π_i and S_λ are considered as vectors, is a module for this induced representation. Thus the Young subgroup, S_λ, leads to a representation of the symmetric group corresponding to the partition λ. Thus tableaux give a visual depiction of symmetric group bases corresponding to various partitions. The irreducible representations of S_n can be labelled by partitions of n. The number of Young tableaux for a given irreducible representation is equal to the dimension of the irreducible representation. Hence, each basis vector is associated with a unique tableau.

In this way, bases of irreducible representations of the direct product subgroup $S_a \times S_b$ are associated with products of tableaux, the first tableau on a symbols, and the second tableau on b symbols.

A more general problem considers irreducible representations of products of more than two tableaux and products of more that two subgroups. Thus we would consider the direct product subgroup

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k},$$

where $n_1 + n_2 + \cdots + n_k = n$. The basis vectors are associated with the product of tableaux,

$$t_1 \times t_2 \times \cdots \times t_k,$$

where t_i is a tableau on n_i symbols.

In this way, we come to the most general form of the transition matrix. This matrix transforms between a subgroup

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k},$$

and a subgroup

$$S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k},$$

where $\sum_i n_i = \sum_j m_j = n$. Associated with the basis vectors are the tableau products

$$t_1 \times t_2 \times \cdots \times t_k \quad \text{and} \quad s_1 \times s_2 \times \cdots \times s_k.$$

Much work has already been done in this area by Hamel et al. [7], McAven et al. [8][9]. In the first paper [7] they consider the matrix which transforms between the Young-

Yamanouchi basis and its dual. The dual basis corresponds to the subgroup chain

$$S_1 \times S_{n-1} \supset S_1 \times S_1 \times S_{n-2} \supset \cdots \supset S_1 \times S_1 \times \cdots S_1.$$

This corresponds to removing the lowest nodes successively from the tableau, or the lowest S_1 factor. The resulting products of skew tableaux are justified using jeu de taquin to give products of normal tableaux. They derive the matrix which transforms between the Young-Yamanouchi basis and the dual basis.

In further papers McAven et al. [8, 9] proceed to the more general problem we have introduced. This begins with the symmetric group S_n from which a subgroup S_{n_1} is removed. The elements of S_{n_1} may be chosen arbitrarily. This corresponds to the arbitrary removal of n_1 symbols from the Young tableau corresponding to the partition λ. Next, the subgroup S_{n_2} is removed, corresponding to the arbitrary removal of n_2 symbols from the tableau. This process is continued until a product of tableaux, corresponding to the product subgroup is obtained. The authors refer to the resulting bases as split bases. They derive the transformation matrix for the product subgroup $S_a \times S_b$ where $b = 3$. This corresponds to the arbitrary removal of three symbols from the tableau.

In this thesis, the author endevors to continue this research in order to find a general form of the transition matrix between different split bases. To do this, we consider several different approaches to the problem.

One approach entails decompositions of partitions restricted to direct product subgroups as given by Robinson [5]. These decompositions are based on group character theory. For example, consider irreducible representation of S_6 corresponding to the partition $(3,2,1)$. First, restrict this representation to the direct product subgroup $S_3 \times S_1 \times S_1 \times S_1$. Using group character theory, Robinson [5] derives the decomposition of this representation as

$$\begin{aligned}[3,2,1] \downarrow S_3 \times S_1 \times S_1 \times S_1 &= [3] \times [2] \times [1] \oplus [3] \times [1^2] \times [1] \oplus 3[2,1] \times [2,1-1] \times [1] \\ &\quad \oplus [1^3] \times [2] \times [1] \oplus [1^3] \times [1^2] \times [1],\end{aligned}$$

with associated skew shapes. Robinson [5] derives similar decompositions for the restriction to other direct product subgroups, for example,

$$[3,2,1] \downarrow S_3 \times S_3.$$

Robinson [5] demonstrates that such a restricted representation gives rise to a resulting representation which is a sum of direct products, with associated direct products of skew shapes. Since the result is a direct product, the use of inner products of characters is appropriate here.

The research conducted by Hamel et al. [7], and McAven et al. [8] emphasizes matrix representations of the symmetric group. Another approach is based on modular representations of the symmetric group. There are several modular representations of the symmetric group. There are the permutation representation, the regular representation (also known as the group algebra), the coset representation and the defining representation.

Another approach to the representation theory of the symmetric group is to use symmetric functions.

A tensor product of representations can be decomposed into irreducible representations by looking at the corresponding product of Schur functions. In this way, the Young subgroup

$$S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell},$$

and its corresponding product of representations can be decomposed into irreducibles.

We have already stated that standard tableaux enable us to construct basis vectors for the irreducible representations of the symmetric group. Thus the problem can be approached combinatorially. In fact, the same graphical method used to find the dimensions of the irreducible representations can be applied to find the characters. Thus there is a link between combinatorial techniques and the decomposition given by Robinson [5] which was based on group theory. An investigation of combinatorial methods and the associated group characters can be used to further the research of Hamel et al. [7], and McAven et al. [8].

The Rachah-Wigner algebra categorises Wigner operators in algebraic terms. The Wigner operator associated with an angular momentum generator is is normalised so that it is bounded. The elements of the Racah-Wigner (R-W) algebra are unit tensor operators known as Wigner operators. The Wigner operators inherit from the Hilbert space structure of quantum physics the properties of normal algebra. The R-W algebra fits into the general framework.

The Racah-Wigner algebra can express the direct product of a number of irreducible representations. As such, it is capable of dealing mathematically with the direct product of irreducible representations. associated with teh direct product of subgroups of the form

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}.$$

The Racah-Wigner algebra has the following properties:-

1. it is a ring
2. it is a linear vector space with multiplication by the complex numbers
3. a norm is defined on the R-W algebra
4. the ring is complete.

The R-W algebra is a graded algebra. As such, it may be possible to bring ring theory to bear on the problem of representation theory investigated in this thesis.

So it is possible to bring a variety of approaches to bear on furthering the research of Hamel et al. [7], and McAven et al. [8]. Since their research is based on matrix representations, it is appropriate to continue with this method until the need for the more powerful technique of modular representation arises. Hamel et al. [7], and McAven et al. [8] also make extensive use of combinatorial tools such as tableaux and associated algorithms. The focus of much of the research in this thesis is to continue to develop and apply these tools.

We have highlighted the link between combinatorial tools and the character theoretic decomposition of Robinson [5]. To explore this link, a formal mathematical approach to the decomposition of tableau is undertaken. The combination of combinatorial and character theoretic techniques is seen as vital to this research, and is the main thrust of this research.

The use of symmetric functions provides a powerful way of researching the representation theory of the symmetric group. Software exists in the public domain to assist in this. This approach is seen as a useful adjunct to the combinatorial approach.

We assume knowledge of group theory, particularly the symmetric group, and of group representations theory. We begin in Chapter 2 with a survey of the representation theory of the symmetric group. We also assume prior knowledge of the following concepts:-

1. Elementary group theory, including group axioms, symmetric groups, subgroups, cosets, group generators, the direct product, normal subgroups, and conjugacy classes ([6], [10]).

2. Linear representation theory, that is, representation of a group by a set of matrices ([4], [12]).

3. Modular representation theory, that is, representation of a group by a module ([4]).

4. Reducible and irreducible representations ([4], [5], [12]).

5. Restricted and induced representations ([4], [5]).

6. Group character theory, including inner products of characters and orthogonality of characters ([4], [5], [19], [20]).

7. Tensor products of representations of subgroups ([4], [5], [19]).

8. The group algebra ([4]).

9. Maschke's Theorem ([4]).

In Chapter 2, we introduce Young subgroups and tableaux. The tabloid expression of subgroups is defined. Standard tableaux are discussed.

Chapter 3 introduces combinatorial techniques for the representation theory of the symmetric group. The all important jeu de taquin is defined. Other results, including the Robinson-Schensted algorithm, are included here.

In Chapter 4, we give a survey of the theory of symmetric functions. The Littlewood-Richardson rule is given here.

In Chapter 5, we give an introduction to the theory of the matrix which transforms between symmetric group bases.

In Chapter 6 the Young theorem for skew tableaux of the general linear group is discussed. Most importantly, we give Robinson's decomposition of tableaux into skew tableaux. Associated with Robinson's decomposition is the theory of lattice permutations. In this chapter, we cover further results on skew diagrams and skew representations.

In Chapter 7 we give a review of the research undertaken by Hamel et al. [7], and McAven et al. [8]. This begins with the transformation between the Young-Yamanouchi basis and its dual. Then a linear equation method for determining multiplicity separation in symmetric group transformation coefficients is discussed. Finally the problem of transforming between split-bases is introduced.

In Chapter 8 we begin our research into transforming between split-bases for the symmetric group. We consider the decomposition of a tableau into a direct product of skew tableau. We develop a formal mathematical treatment of such decomposition. In chapter 9 we then develop an equivalent mathematical treatment of the decomposition of a tableau into a cartesian product of normal tableau.

In Chapter 10 we give a concluding survey of the research performed thus far.

Chapter 2

Representations of the Symmetric Group

The research by Hamel et al.[7], and McAven et al.[8] in Chapter 7 invokes the Littlewood-Richardson rule. In Chapter 4 we introduce symmetric functions in sufficient detail to state the Littewood-Richardson rule. In this chapter we develop the machinery needed in Chapter 4.

Significantly, we state Young's rule, which is used in the development of the Littlewood-Richardson rule. This requires that we first define tableaux, Young tableaux, and standard tableaux. We then define tabloids in terms of tableaux. This leads to the definition of polytabloids.

The Specht module is defined in terms of polytabloids. We then define generalised tableaux and semi-standard tableaux. This leads to the definition of Kosta numbers. We define the permutation module. Finally we use the definitions of the permutation module, the Specht module, and Kostka numbers to state Young's rule. This limited coverage closely follows Sagan's[4] text.

In this chapter we deliberately keep theorems to the minimum. Instead, we focus primarily on providing the definitions needed to state Young's rule. Importantly this chapter also serves the purpose of introducing definitions which are used in later chapters and in the author's research, for example, tableaux, Young tableaux and standard tableaux.

We wish to construct all of the irreducible representations of the symmetric group. The number of irreducible representations is equal to the number of conjugacy classes. In the case of the symmetric group S_n, this is the number of partitions of n. For each partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, there is a corresponding subgroup S_λ which is an isomorphic copy of $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}$ inside S_n.

2.1 Young Subgroups, Tableaux, and Tabloids

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ of n is denoted by $\lambda \vdash n$. It is clear that $\mid \lambda \mid = \sum_i \lambda_i$, so that a partition of n satisfies $\mid \lambda \mid = n$.

We begin our definition of tableaux by defining a Ferrers diagram.

Definition 2.1: Suppose that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a partition of n. The Ferrers diagram, or shape, of λ is an array of n dots or cells in left-justified rows with row i containing λ_i dots for $1 \leq i \leq l$. The dot in row i and column j has coordinates (i,j).

Example 2.1 *The partition $\lambda = (3,3,2,1)$ of 9 has Ferrers diagram as shown below*

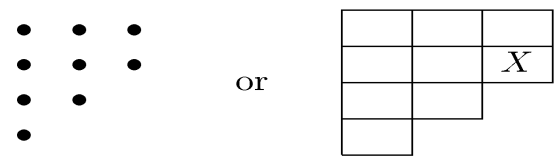


Figure 2.1: Ferrers diagram

where the box in the $(2,3)$ position has an X in it. The cell marked X has coordinates $(2,3)$.

Definition 2.2: Suppose that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a partition of n. The corresponding Young subgroup of S_n is

$$S_\lambda = S_{\{1,2,\cdots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\cdots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_l+1,n-\lambda_l+2,\cdots,n\}}.$$

Example 2.2

$$\begin{aligned} S_{(3,3,2,1)} &= S_{\{1,2,3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9\}} \\ &\cong S_3 \times S_3 \times S_2 \times S_1. \end{aligned}$$

In general, $S_{(\lambda_1,\lambda_2,\cdots,\lambda_l)}$ and $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}$ are isomorphic as groups.

The definition of a Ferrers diagram leads to the definition of a Young tableau.

Definition 2.3 A Young tableau of shape λ, t, is an array obtained by replacing the dots of the Ferrers diagram of λ with the numbers $1, 2, \cdots, n$ bijectively. A Young tableau of shape λ is also called a λ-tableau and denoted by t^λ. We may also write that the shape of the tableau, sh $t = \lambda$.

We denote Young tableaux with lowercase letters.

Example 2.3 *For the shape $\lambda = (2,1)$, the list of all possible Young tableaux of shape λ is*

$$t: \begin{matrix} 1 & 2 \\ 3 & \end{matrix}, \begin{matrix} 2 & 1 \\ 3 & \end{matrix}, \begin{matrix} 1 & 3 \\ 2 & \end{matrix}, \begin{matrix} 3 & 1 \\ 2 & \end{matrix}, \begin{matrix} 2 & 3 \\ 1 & \end{matrix}, \begin{matrix} 3 & 2 \\ 1 & \end{matrix}.$$

In the following discussion the term "tableau" will refer to a Young tableau unless otherwise stated. Two λ-tableaux t_1 and t_2 are row equivalent, $t_1 \sim t_2$, if corresponding rows of the two tableaux contain the same elements.

The statement of row-equivalent tableaux leads to the definition of a tabloid.

Definition 2.4: A tabloid of shape λ, or a λ-tabloid, is

$$\{t\} = \{t_1 \mid t_1 \sim t\} \quad \text{where } \mathrm{sh}\, t = \lambda.$$

Example 2.4 *For the tableaux*

$$t = \begin{matrix} 1 & 2 \\ 3 & \end{matrix},$$

the corresponding tabloid is

$$\{t\} = \left\{ \begin{matrix} 1 & 2 \\ 3 & \end{matrix}, \begin{matrix} 2 & 1 \\ 3 & \end{matrix} \right\} = \begin{matrix} \overline{\underline{1 \quad 2}} \\ \underline{3} \end{matrix}.$$

For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \vdash n$, the number of tableaux in any equivalence class is $\lambda_1!\lambda_2!\cdots\lambda_l! \stackrel{\text{def}}{=}$

$\lambda!$. The number of λ-tabloids is then $n!/\lambda!$

A permutation $\pi \in S_n$ acts on a tableau $t = (t_{i,j})$ of shape $\lambda \vdash n$ in the following manner:

$$\pi t = (\pi(t_{i,j})).$$

Example 2.5

$$(1,2,3) \begin{array}{cc} 1 & 2 \\ 3 & \end{array} = \begin{array}{cc} 3 & 1 \\ 2 & \end{array}.$$

This action on tabloids is described by

$$\pi\{t\} = \{\pi t\}.$$

That is, a permutation acts on a tableau by permuting the elements of the tableau. As this action is independent of the choice of t, it is well defined. This action is associated with an S_n-module.

The permutation module is defined in terms of tabloids.

Definition 2.5: Suppose that $\lambda \vdash n$. We define the M-module to be

$$M^{\lambda} = \mathbb{C}\{\{\boldsymbol{t}_1\}, \{\boldsymbol{t}_2\}, \cdots, \{\boldsymbol{t}_k\}\},$$

where $\{\boldsymbol{t}_1\}, \{\boldsymbol{t}_2\}, \cdots, \{\boldsymbol{t}_k\}$ is a complete list of λ-tabloids. The module M^{λ} is called the permutation module corresponding to λ. That is, M^{λ} is the module spanned by the λ-tabloids. The permutation module is a special case of an S_n-module, obtained from a linear combination of the list of λ-tabloids.

Example 2.6 *If $\lambda = (n)$, then*

$$M^{(n)} = \mathbb{C}\{\overline{\underline{\mathbf{12}\cdots\boldsymbol{n}}}\},$$

with the trivial action.

Example 2.7 *Suppose that $\lambda = (1^n)$. Each equivalence class consists of a single tableau. Each tableau is associated with a permutation in one-line notation. The action of S_n is preserved, giving rise to the module*

$$M(1^n) \cong \mathbb{C}[S_n].$$

This is the regular representation.

Example 2.8 *Consider these modules for the case $n = 3$. The full set of partitions is $\lambda = (3), (2,1)$, and (1^3). The associated modules correspond to the trivial, defining and regular representations respectively. Let the character of M^λ be ϕ^λ, and the conjugacy class of S_3 corresponding to μ be K_μ. This gives rise to the following character table:*

	$K_{(1)}$	$K_{(2,1)}$	$K_{(3)}$
$\phi^{(3)}$	1	1	1
$\phi^{(2,1)}$	3	1	0
$\phi^{(1^3)}$	6	0	0

Any G-module is cyclic if there is a $\boldsymbol{v} \in M$ such that

$$M = \mathbb{C}G\boldsymbol{v} \qquad \text{and} \qquad G\boldsymbol{v} = \{g\boldsymbol{v} \mid g \in G\}.$$

Such a module M is generated by $\boldsymbol{v}$. M^λ is cyclic because any λ-tabloid can be transformed to any other tabloid of the same shape by some permutation. If $\lambda \vdash n$, then M^λ is cyclic, generated by any given λ-tabloid. The dimension is given by $\dim M^\lambda = n!/\lambda!$, where $\dim M^\lambda$ is the number of λ-tabloids.

The symmetric group

$$S_\lambda = S_{\{1,2,\cdots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\cdots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_l+1,n-\lambda_l+2,\cdots,n\}},$$

is modelled by the tabloid

$$\{t^\lambda\} = \begin{array}{cccc} \hline 1 & 2 & \cdots & \lambda_1 \\ \hline \lambda_1+1 & \lambda_1+2 & \cdots & \lambda_1+\lambda_2 \\ \hline & & \vdots & \\ \hline n-\lambda_l+1 & \cdot & \cdots & n \\ \hline \end{array} .$$

The order of these integers is immaterial in t^λ, as they all occur in the same row. The coset πS_λ corresponds to the tabloid $\{\pi t^\lambda\}$.

A standard tableau is a special case of a Young tableau, as in the following definition.

Definition 2.6: A tableau t is standard if the rows and columns of t are increasing sequences.

Example 2.9 *The tableau*

$$t = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array}$$

is standard, but

$$t = \begin{array}{ccc} 1 & 2 & 3 \\ 6 & 4 & \\ 5 & & \end{array}$$

is not.

Definition 2.7: A composition of n is an ordered sequence of non-negative integers $\mu = (\mu_1, \mu_2, \cdots, \mu_l)$ such that $\sum_i \mu_i = n$. The integers μ_i are called the parts of the composition.

2.2 Specht Modules

The Specht modules are all of the irreducible modules. To define the Specht module, we must first define the row-stabilizer and column stabilizer.

Definition 2.8: Suppose that a tableau t has rows $R_1, R_2, \cdots, R_l$ and columns $C_1, C_2, \cdots, C_k$.

Then

$$R_t = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l},$$

is the row-stabilizer of t, and

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k},$$

is the column-stabilizer of t.

In this formulation, S_{R_i} and S_{C_i} represent symmetric subgroups on the integers contained in row i and column j respectively.

Example 2.10 *Consider the tableau*

$$t = \begin{array}{ccc} 4 & 1 & 2 \\ 3 & 5 & \end{array}.$$

Then $R_t = S_{\{1,2,4\}} \times S_{\{3,5\}}$ *and* $C_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}}$.

The equivalence classes can be expressed as $\{t\} = R_t t$. These groups are associated with the elements of $\mathbb{C}[S_n]$ where $\mathbb{C}[S_n]$ is the G-module associated with S_n. For any subset $H \subseteq S_n$, the group algebra sums are

$$H^+ = \sum_{\pi \in H} \pi,$$

and

$$H^{-} = \sum_{\pi \in H} \mathrm{sgn}(\pi)\pi.$$

The value of κ_t is

$$\kappa_t \stackrel{\mathrm{def}}{=} C_t^{-} = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi.$$

If t has columns $C_1, C_2, \cdots, C_k$, then κ_t can be expressed as

$$\kappa_t = \kappa_{C_1} \kappa_{C_2} \cdots \kappa_{C_k}.$$

That is, κ_t is the product of the group algebra sums of the columns of the tableaux.

Having defined κ_t in terms of the column stabilizer, we can now define the polytabloid in terms of κ_t and the tabloid.

Definition 2.9: If t is a tableau, then the associated polytabloid is

$$e_t = \kappa_t \{\boldsymbol{t}\}.$$

Example 2.11 *For*

$$t = \begin{matrix} 4 & 1 & 2 \\ 3 & 5 & \end{matrix},$$

we have that $C_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}}$.

Hence $\kappa_t = C_t = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi = \sum_{\pi \in \{3,4\}} \mathrm{sgn}(\pi)\pi \times \sum_{\pi \in \{1,5\}} \mathrm{sgn}(\pi)\pi \times \sum_{\pi \in \{2\}} \mathrm{sgn}(\pi)\pi.$

This gives

$$\begin{aligned}
\kappa_t &= (e-(3,4))(e-(1,5)), \\
&= e-(3,4)e-(1,5)e+(3,4)(1,5),
\end{aligned}$$

giving

$$e_t = \begin{array}{l} \hline 4\ 1\ 2 \\ \hline 3\ 5 \\ \hline \end{array} - \begin{array}{l} \hline 3\ 1\ 2 \\ \hline 4\ 5 \\ \hline \end{array} - \begin{array}{l} \hline 4\ 5\ 2 \\ \hline 3\ 1 \\ \hline \end{array} + \begin{array}{l} \hline 3\ 5\ 2 \\ \hline 4\ 1 \\ \hline \end{array}.$$

Theorem 2.1 *From Sagan[4] (page 61) let t be a tableau and π be a permutation. We have that $e_{\pi_t} = \pi_{e_t}$.*

This theorem states that the action of a permutation on a tableau is to permute the elements of the associated polytabloid.

Having defined a polytabloid, we can now define the Specht module.

Definition 2.10: For any partition λ, the corresponding Specht module, S^λ, is the submodule of M^λ spanned by the polytabloids e_t, where t is of shape λ. The S^λ are cyclic modules generated by any given polytabloid.

Example 2.12 *If $\lambda = (n)$, then $e_{12\cdots n} = \begin{array}{l} \hline 12\cdots n \\ \hline \end{array}$ is the only polytabloid and $S^{(n)}$ carries the trivial representation. $S^{(n)}$ is a submodule of $M^{(n)}$ where S_n acts trivially.*

Example 2.13 *Let $\lambda = (1^n)$ and*

$$t = \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array}.$$

The column stabilizer is $S_{\{1,2,\cdots,n\}}$, the symmetric group on n elements. This gives that $\kappa_t = \sum_{\sigma\in S_n} (\mathrm{sgn}\sigma)\sigma$. Then e_t is the signed sum of all $n!$ permutations regarded as tabloids. For any permutation π, we have that

$$e_{\pi t} = \pi e_t = \sum_{\sigma\in S_n} (\mathrm{sgn}\sigma)\pi\sigma\{\boldsymbol{t}\}.$$

Replacing $\pi\sigma$ by τ gives

$$e_{\pi t} = \sum_{\tau\in S_n} (\mathrm{sgn}\pi^{-1}\tau)\tau\{\boldsymbol{t}\} = (\mathrm{sgn}\pi^{-1}) \sum_{\tau\in S_n} (\mathrm{sgn}\tau)\tau\{\boldsymbol{t}\} = (\mathrm{sgn}\pi)e_t,$$

as $\mathrm{sgn}\pi^{-1} = \mathrm{sgn}\pi$. This means that every polytabloid is a scalar multiple of e_t. This gives that

$$S(1^n) = \mathbb{C}\{e_t\},$$

with the action that $\pi e_t = (\mathrm{sgn}\pi)e_t$. This is the sign representation.

Example 2.14 *For the partition $\lambda = (n-1,1) \vdash n$, the corresponding tabloids are*

$$\{t\} = \begin{array}{c} \hline \boldsymbol{i} \ \cdots \ \boldsymbol{j} \\ \hline \boldsymbol{k} \\ \hline \end{array} \stackrel{\text{def}}{=} \boldsymbol{k},$$

where $1 \le k \le n$. In this example, the tabloids are completely determined by the element *in the second row. We have that*

$$\begin{aligned} e_t &= \kappa_t\{\boldsymbol{t}\} \\ &= \textstyle\sum_{\pi\in C_t}(\operatorname{sgn}\pi)\pi\{\boldsymbol{t}\} \\ &= \textstyle\sum_{\pi\in S_{i,k}}(\operatorname{sgn}\pi)\pi\{\boldsymbol{t}\} \\ &= (e-(i,k))\{\boldsymbol{t}\} \end{aligned}$$

For this tabloid, $e_t = \boldsymbol{k} - \boldsymbol{i}$*, giving*

$$S^{(n-1,1)} = \{c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\boldsymbol{n} \mid c_1 + c_2 + \cdots + c_n = 0\}.$$

The dimension of this representation is $\dim S^{(n-1,1)} = n-1$.

2.3 Inner and Outer Corners

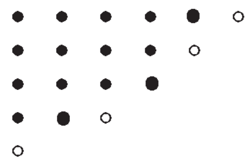
We wish to consider the restriction and induction of an irreducible representation for S^λ. We begin by defining an inner corner of a Ferrers diagram.

Definition 2.11: If λ is a Ferrers diagram, then an inner corner of λ is a node $(i, j) \in \lambda$ whose removal leaves a Ferrers diagram. The set of partitions obtained by the removal of the node is denoted by λ^-.

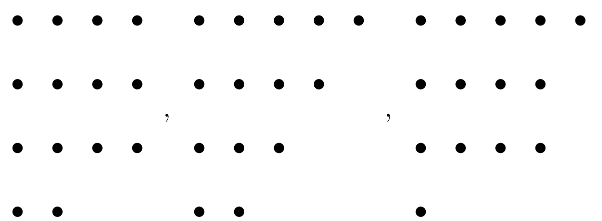
Also, we must define an outer corner of a Ferrers diagram.

Definition 2.12: An outer corner of λ is a node $(i, j) \notin \lambda$ whose addition fixes a Ferrers diagram. The set of partitions thus obtained is denoted by λ^+.

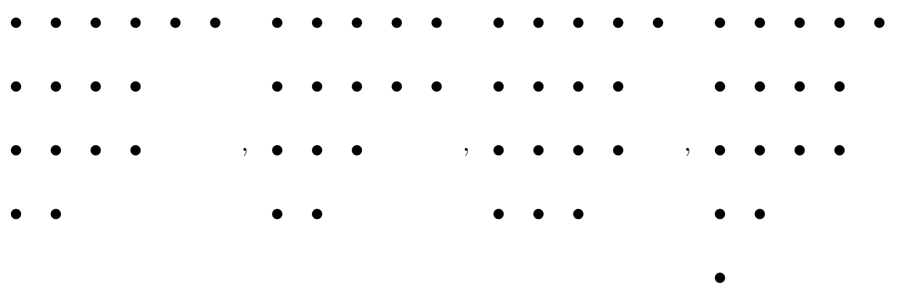
The inner corners of λ are those nodes at the end of a row and column of λ. For example, if $\lambda = (5,4,4,2)$, then the inner corners are enlarged and the outer corners are marked with open circles in the diagram below:

$\lambda =$ 

So after removal, we could have that

$\lambda^- :$ 

whereas after addition, the possibilities are

$\lambda^+ :$ 

These are the partitions which occur in restriction and induction. We have that

$$S^{(5,4,4,2)} \downarrow_{S_{14}} \cong S^{(4,4,4,2)} \oplus S^{(5,4,3,2)} \oplus S^{(5,4,4,1)},$$

and

$$S^{(5,4,4,2)} \uparrow^{S_{16}} \cong S^{(6,4,4,2)} \oplus S^{(5,5,4,2)} \oplus S^{(5,4,4,3)} \oplus S^{(5,4,4,2,1)}.$$

2.4 Kostka Numbers and Young's Rule

We wish to state Young's rule which is used later to formulate the Littlewood-Richardson rule. This requires some definitions.

Definition 2.13: A generalized Young tableau of shape λ, T, is obtained by putting positive integers in the nodes of λ, with repetitions allowed. The type or content of T is the composition

$$\mu = (\mu_1, \mu_2, \cdots, \mu_m),$$

where μ_i equals the number of i's in T. We write

$$T_{\lambda\mu} = \{T \mid T \text{ has the shape } \lambda \text{ and content } \mu\}.$$

Capital letters are used to denote generalized tableaux.

Example 2.15 *The generalized tableau*

$$T = \begin{matrix} 4 & 1 & 4 \\ 1 & 3 & \end{matrix},$$

has shape $\lambda = (3,2)$ *and content* $\mu = (2,0,1,2)$.

A semi-standard tableau is a special case of a generalized tableau, as in the following definition.

Definition 2.14: A generalized tableau is semi-standard if its rows are weakly increasing and its columns are strictly increasing. The set of semi-standard λ-tableau of type μ is denoted by $\tau^0_{\lambda\mu}$.

Example 2.16 *The tableau*

$$S = \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \end{matrix},$$

is semi-standard, but the tableau

$$T = \begin{matrix} 2 & 1 & 1 \\ 3 & 2 & \end{matrix},$$

is not.

The Kostka numbers are

$$K_{\lambda\mu} = \mid \tau^0_{\lambda\mu} \mid .$$

That is, the Kosta numbers are the number of semi-standard tableaux having shape λ and content μ.

Theorem 2.2 ***Young's Rule, Sagan [4], page 85***

The multiplicity of S^λ in M^μ is equal to the number of semi-standard tableaux of shape λ

and content μ, that is,

$$M^{\mu} \cong \bigoplus_{\lambda} K_{\lambda\mu} S^{\lambda}.$$

Example 2.17 *Suppose that $\mu = (2,2,1)$. Then*

$$M^{(2,2,1)} \cong S^{(2,2,1)} \oplus S^{(3,3,1)} \oplus 2S^{(2,2)} \oplus 2S^{(4,1)} \oplus S^{(5)}.$$

This theorem shows that the Kosta numbers relate the Specht module to the permutation module.

Example 2.18 *For any μ, $K_{\mu\mu} = 1$, because the only μ-tableau of content μ is the one with all the 1's in row 1, all the 2's in row 2, etc.*

Example 2.19 *For any μ, $K_{(n)}\mu = 1$, because there is only one way to arrange the numbers in weakly increasing order.*

Example 2.20 *For any λ, $K_{\lambda(1^n)} = f^{\lambda}$, the number of standard tableaux of shape λ. This means that*

$$M^{(1^n)} \cong \bigoplus_{\lambda} f^{\lambda} S^{\lambda}.$$

$M^{(1^n)}$ is the regular representation, and $f^{\lambda} = \dim S^{\lambda}$.

Chapter 3

Combinatorial Algorithms

In Chapter 7 we survey the research of Hamel et al.[7], and McAven et al.[8]. They apply combinatorial operations on tableaux to the representation theory of the symmetric group. In particular, they apply the technique of jeu de taquin. We define this technique in this chapter.

In furthering the research on Hamel et al. [7], and McAven et al. [8], undertaken in Chapters 8 and 9, we make use of the Robinson-Schensted algorithm. We also make use of the concept of Knuth-equivalence of tableaux. In this chapter we define and discuss these concepts.

This limited coverage closely follows the book by Sagan[4]. For a more comprehensive coverage, the interested reader is referred to the books by Stanley[15],[16] and Stanton and White[18].

Representations of the symmetric group can be developed using combinational techniques. This is because the number of standard Young tableaux of a given shape is the degree of the corresponding representation.

3.1 The Hook Formula

There exists a simple formula for the number of standard tableaux of shape λ. This involves the concept of the hook.

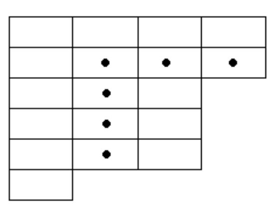
Definition 3.1: If $v = (i, j)$ is a node in the diagram of λ, then it has a hook,

$$H_v = H_{i,j} = \{(i, j') \mid j' \geq j\} \cup \{(i', j) \mid i' \geq i\},$$

with corresponding hooklength

$$h_v = h_{i,j} = \mid H_{i,j} \mid .$$

Example 3.1 *Given* $\lambda = (4^2, 3^3, 1)$*, then the node* $(2,2)$ *has a hook with hooklength* $h_{2,2} =$



6 *as shown in the diagram.*

It turns out that the number of standard λ-tableaux can be expressed in terms of the hooklengths of each cell in the λ-tableau.

Theorem 3.1 ***Hook Formula, Sagan[4] (page 124)***

If $\lambda \vdash n$, then

$$f^{\lambda} = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

This theorem is important as it establishes the number of standard tableaux of shape λ.

Example 3.2 *Given $\lambda = (2,2,1) \vdash 5$, the hooklengths are given in the array*

4	2
3	1
1	

where $h_{i,j}$ is placed in cell (i,j). Therefore

$$f^{(2,2,1)} = \frac{5!}{4.3.2.1^2} = 5.$$

This result is illustrated by listing the tableaux:

$$\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array}, \quad \begin{array}{cc} 1 & 2 \\ 3 & 5 \\ 4 & \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 2 & 5 \\ 4 & \end{array}, \quad \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}.$$

3.2 The Robinson-Schensted Algorithm

In studying the degree of a representation, we will make use of the Robinson-Schensted algorithm. We have the following identity.

Theorem 3.2 *(Sagan [4], page 91)*

$$\sum_{\lambda \vdash n} (f^{\lambda})^2 = n!.$$

This is a corollary of Maschke's Theorem. This theorem says that the number of elements in S_n is equal to the number of pairs of standard tableaux of the same shape λ as λ varies over all partitions of n. This bijection is denoted by $\pi \overset{R-S}{\leftrightarrow} (s,t)$ where $\pi \in S_n$, and s,t are standard λ-tableaux as produced by the Robinson-Schensted algorithm.

The Robinson-Schenested algorithm may be applied to partial tableaux defined as follows.

Definition 3.2: A tableau is a partial tableau if its rows and columns increase. Hence a partial tableau will be a standard tableau if its elements are the set $\{1, 2, \cdots, n\}$.

Example 3.3 *The tableau*

$$t = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \quad .$$

is a partial tableau.

Given a permutation π in two-line notation

$$\pi = \begin{matrix} 1 & 2 & \cdots & n \\ \\ x_1 & x_2 & \cdots & x_n \end{matrix},$$

we construct a sequence of tableaux

$$(s_0, t_0) = (\phi, \phi), (s_1, t_1), (s_2, t_2), \cdots, (s_n, t_n) = (s, t),$$

where $x_1, x_2, \cdots, x_n$ are inserted into the s tableau and $1, 2, \cdots, n$ are placed in the t tableau, so that sh $s_k =$ sh t_k for all k.

The operations of insertion and placement are described by the Robinson-Schensted algorithm.

Suppose s is a partial tableau. Then row insertion of x into s is as follows:

Algorithm R-S. Let s be a partial tableau. Also, let x be an element not in s. To row insert x into s, we proceed as follows (where := means replacement).

RS1: Set $R :=$ the first row of s.

RS2: **While** x is less than some element of row R, **do**

> **(RSa)** Let y be the smallest element of R greater than x and replace y by x in R (denoted by $R \leftarrow x$).
>
> **(RSb)** Set $x := y$ and $R :=$ the next row down.

RS3: Now x is greater than every element of R, so place x at the end of row R and **stop**.

This process is illustrated below.

Example 3.4 *To illustrate, suppose* $x = 3$ *and*

$$s = \begin{array}{cccc} 1 & 2 & 5 & 8 \\ 4 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} .$$

To follow the path of the insertion of x into s, we put elements that are displaced (or bumped) during the insertion in boldface type:

$$\begin{array}{cccc} 1 & 2 & 5 & 8 \\ 4 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} \leftarrow \mathbf{3} \quad \begin{array}{cccc} 1 & 2 & \mathbf{3} & 8 \\ 4 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} \begin{array}{c} \\ \leftarrow \mathbf{5} \\ \\ \\ \end{array} \quad \begin{array}{cccc} 1 & 2 & 3 & 8 \\ 4 & 5 & & \\ 6 & & & \\ 9 & & & \end{array} \begin{array}{c} \\ \\ \leftarrow \mathbf{7} \\ \\ \end{array} \quad \begin{array}{cccc} 1 & 2 & 3 & 8 \\ 4 & 5 & & \\ 6 & \mathbf{7} & & \\ 9 & & & \end{array} .$$

Since the result of row inserting x into tableau s yields the tableau s', we write

$$r_x(s) = s'.$$

The insertion rules ensure that the tableau s' has increasing rows and columns.

For the placement of an element in a tableau, suppose that t is a partial tableau of shape μ, and that (i, j) an outer corner of μ. If k is greater than every element of t then to place k in t at position (i, j), set $t_{i,j} := k$. That is, we place k at the selected outer corner of t. As

k is greater than every element of t, t' is still a partial tableau.

Example 3.5 *Let*

$$t = \begin{array}{lll} 1 & 2 & 5 \\ 4 & 7 & \\ 6 & & \\ 8 & & \end{array},$$

then placing $k = 9$ *in cell* $(i, j) = (2,3)$ *yields*

$$\begin{array}{lll} 1 & 2 & 5 \\ 4 & 7 & 9 \\ 6 & & \\ 8 & & \end{array}.$$

To build the sequence of tableau pairs from the permutation π, we first start with a tableau pair (s_0, t_0) of empty tableaux. Assuming that (s_{k-1}, t_{k-1}) has been built, we then define the tableau pair (s_k, t_k) by

$$\begin{aligned} s_k &= r_{x_k}(s_{k-1}), \text{ and} \\ t_k &= \text{place } k \text{ into } t_{k-1} \text{ at the position } (i, j) \\ &\quad \text{ where the insertion terminated.} \end{aligned}$$

The operation of placement into t_k ensures that sh $s_k =$ sh t_k for all k. We denote the tableau $s = s_n$ by the s-tableau, or insertion tableau, of π and write $s = s(\pi)$. Similarly, the tableau $t = t_n$ is called the t-tableau, or recording tableau, denoted by $t = t(\pi)$.

Example 3.6 *Now we consider an example of the complete algorithm. Boldface numbers are used for the elements of the lower line of* π *and hence also for the elements of the* s_k*.*

Let

$$\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{4} & \mathbf{2} & \mathbf{3} & \mathbf{6} & \mathbf{5} & \mathbf{1} & \mathbf{7} \end{matrix}.$$

Then the tableaux constructed by the algorithm are

$$\boldsymbol{s}_k: \ \phi,\quad \mathbf{4},\quad \begin{matrix}\mathbf{2}, \\ \mathbf{4}\end{matrix}\quad \begin{matrix}\mathbf{2} & \mathbf{3} \\ \mathbf{4} & \end{matrix},\quad \begin{matrix}\mathbf{2} & \mathbf{3} & \mathbf{6} \\ \mathbf{4} & & \end{matrix},\quad \begin{matrix}\mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{4} & \mathbf{6} & \end{matrix},\quad \begin{matrix}\mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{2} & \mathbf{6} & \\ \mathbf{4} & & \end{matrix},\quad \begin{matrix}\mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} \\ \mathbf{2} & \mathbf{6} & & \\ \mathbf{4} & & & \end{matrix} = \boldsymbol{s},$$

$$t_k: \ \phi,\quad 1,\quad \begin{matrix}1, \\ 2\end{matrix}\quad \begin{matrix}1 & 3 \\ 2 & \end{matrix},\quad \begin{matrix}1 & 3 & 4 \\ 2 & & \end{matrix},\quad \begin{matrix}1 & 3 & 4 \\ 2 & 5 & \end{matrix},\quad \begin{matrix}1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{matrix},\quad \begin{matrix}1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix} = t,$$

so

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{4} & \mathbf{2} & \mathbf{3} & \mathbf{6} & \mathbf{5} & \mathbf{1} & \mathbf{7} \end{matrix} \xrightarrow{R-S} \left(\begin{matrix}\mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} \\ \mathbf{2} & \mathbf{6} & & \\ 4 & & & \end{matrix}, \begin{matrix}1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix} \right).$$

In studying the Robinson-Schensted algorithm we require the definition of a row word.

Definition 3.3: If t is a tableau, then we define the row word of t to be the permutation

$$\pi_t = R_l R_{l-1} \cdots R_1,$$

where $R_1, \cdots, R_l$ are the rows of t.

Example 3.7 *Let*

$$t = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix} .$$

Then

$$\pi_t = 4\,2\,6\,1\,3\,5\,7.$$

Lemma 3.1 *(Sagan [4], page 101) If s is a standard tableau, then*

$$\pi_s \stackrel{R-S}{\longrightarrow} (s, \cdot).$$

That is, if s is a standard tableau, then the insertion tableau for the row word of s is the tableau s itself. This result is not significant in itself, but it helps to clarify the statement of Theorem 3.6.

3.3 The Knuth Relations

We are interested in equivalence classes of Robinson-Schensted insertion tableaux of permutations. Here we make the following definitions.

Definition 3.4: If $\mu \subseteq \lambda$ as Ferrers diagrams, then the corresponding skew diagram, or skew shape, or skew tableau, is the set of cells

$$\lambda/\mu = \{c \mid c \in \lambda \text{ and } c \notin \mu\}.$$

A skew diagram is normal if $\mu = 0$.

Example 3.8 *If* $\lambda = (3,3,2,1)$ *and* $\mu = (2,1,1)$, *then we have the skew diagram as shown in Figure 3.1.*

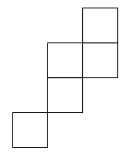
$\lambda/\mu =$  .

Figure 3.1: Example 3.8

Normal shapes are the left-justified tableaux consider so far.

The definitions of skew tableaux, standard skew tableaux and so on, all follow the same conventions as developed so far.

We define an equivalence relation on the group S_n.

Definition 3.5: Two permutations $\pi, \sigma \in S_n$ are said to be s-equivalent, written $\pi \overset{s}{\cong} \sigma$, if $s(\pi) = s(\sigma)$. Note that $S(\pi) = S(\sigma)$ implies that the insertion tableaux corresponding to π and σ are equal. This relation partitions S_n into equivalence classes.

Example 3.9 *The equivalence classes in* S_3 *are*

$$\{1\,2\,3\}, \{2\,1\,3, 2\,3\,1\}, \{1\,3\,2, 3\,1\,2\}, \{3\,2\,1\}$$

corresponding to the insertion tableau

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \quad \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array},$$

respectively.

In order to provide an alternative description of s-equivalence, we define the Knuth relations.

Definition 3.6: Suppose that $x < y < z$. Then $\pi, \sigma \in S_n$ differ by a Knuth relation of the first kind, written $\pi \overset{1}{\cong} \sigma$, if

$$\begin{aligned} \pi &= x_1 \cdots yxz \cdots x_n, \text{ and} \\ \sigma &= x_1 \cdots yzx \cdots x_n, \text{ or vice versa.} \end{aligned}$$

Definition 3.7: The two permutations differ by a Knuth relation of the second kind, written $\pi \overset{2}{\cong} \sigma$, if

$$\begin{aligned} \pi &= x_1 \cdots xzy \cdots x_n, \text{ and} \\ \sigma &= x_1 \cdots zxy \cdots x_n, \text{ or vice versa.} \end{aligned}$$

Definition 3.8: The two permutations are Knuth equivalent, written $\pi \overset{K}{\cong} \sigma$, if there exists a sequence of permutations such that

$$\pi = \pi_1 \overset{i}{\cong} \pi_2 \overset{j}{\cong} \cdots \overset{l}{\cong} \pi_k = \sigma,$$

where $i, j, \cdots, l \in \{1, 2\}$.

Example 3.10 *In the previous example, we see that the only non-trivial Knuth relations are*

$$2\,1\,3 \overset{1}{\cong} 2\,3\,1 \quad \textit{and} \quad 1\,3\,2 \overset{2}{\cong} 3\,1\,2.$$

It can be seen that the Knuth equivalence classes and the s-equivalence classes coincide. This always happens, as seen in the following important theorem.

Theorem 3.3 *(Sagan [4], page 100) If* $\pi, \sigma \in S_n$*, then*

$$\pi \overset{K}{\cong} \sigma \quad \Leftrightarrow \quad \pi \overset{s}{\cong} \sigma.$$

The preceding definitions and theorems involving standard tableaux and permutations are equally valid when applied to partial tableaux and permutations. This is because we may set up a bijection

$$1 \to k_1,\ 2 \to k_2,\ \cdots,\ n \to k_n,$$

between the elements $\{1, 2, \cdots, n\}$ of a standard tableaux and the elements $\{k_1, k_2, \cdots, k_n\}$ of a partial tableau.

3.4 Schutzenberger's Jeu De Taquin

The jeu de taquin of Schutzenberger can be used to justify skew tableaux to normal tableaux.

The definition of jeu de taquin requires the definition of forward and backward slides. The forward slide is defined by Algorithm F. Given a partial tableau t of shape λ/μ, a forward slide on t into a cell c is as follows:

ALGORITHM F

F1: Pick c to be an inner corner of μ.

F2: **While** c is not an inner corner of λ **do**

- **(Fa)** If $c = (i, j)$, then let c' be the cell of $\min\{t_{i+1,j}, t_{i,j+1}\}$.
- **(Fb)** Slide $t_{c'}$ into cell c and let $c := c'$.

If only one of $t_{i-1,j}$, $t_{i,j-1}$ exists in step Fa, then the minimum is taken to be that single value. The resulting tableau is denoted by $j_c(t)$.

Example 3.11 *A forward slide is illustrated below:*

By way of illustration, let

$$t = \begin{array}{cccc} & & 6 & 8 \\ & 2 & 4 & 5 & 9 \\ 1 & 3 & 7 & & \end{array} \cdot$$

We let a dot indicate the position of the empty cell as we perform a forward slide from $c = (1,3)$.

$$\begin{array}{cccc} & & \bullet & 6 & 8 \\ & 2 & 4 & 5 & 9 \\ 1 & 3 & 7 & & \end{array}, \quad \begin{array}{cccc} & & 4 & 6 & 8 \\ & 2 & \bullet & 5 & 9 \\ 1 & 3 & 7 & & \end{array}, \quad \begin{array}{cccc} & & 4 & 6 & 8 \\ & 2 & 5 & \bullet & 9 \\ 1 & 3 & 7 & & \end{array}, \quad \begin{array}{cccc} & & 4 & 6 & 8 \\ & 2 & 5 & 9 & \bullet \\ 1 & 3 & 7 & & \end{array}.$$

Thus

$$j_c(t) = \begin{array}{ccccc} & & 4 & 6 & 8 \\ & 2 & 5 & 9 & \\ 1 & 3 & 7 & & \end{array}.$$

Similarly, a backward slide on t into a cell c is defined as

ALGORITHM B

B1: Pick c to be an outer corner of λ.

B2: While c is not an outer corner of μ **do**

> **(Ba)** If $c = (i, j)$, then let c' be the cell of $\max\{t_{i-1,j}, t_{i,j-1}\}$.
>
> **(Bb)** Slide $t_{c'}$ into cell and let $c := c'$.

This produces a tableau $j^c(t)$.

Example 3.12 *A backward slide is illustrated below:*

A backward slide from $c = (3,4)$ *looks like the following*

$$\begin{array}{ccccc} & & & 6 & 8 \\ & 2 & 4 & 5 & 9 \\ 1 & 3 & 7 & \bullet & \end{array}, \quad \begin{array}{ccccc} & & & 6 & 8 \\ & 2 & 4 & 5 & 9 \\ 1 & 3 & \bullet & 7 & \end{array}, \quad \begin{array}{ccccc} & & & 6 & 8 \\ & 2 & \bullet & 5 & 9 \\ 1 & 3 & 4 & 7 & \end{array}, \quad \begin{array}{ccccc} & & & 6 & 8 \\ & \bullet & 2 & 5 & 9 \\ 1 & 3 & 4 & 7 & \end{array},$$

so

$$j^c(t) = \begin{array}{ccccc} & & & 6 & 8 \\ & & 2 & 5 & 9 \\ 1 & 3 & 4 & 7 & \end{array}.$$

A slide is an invertible operation. If c is a cell for a forward slide on t and the cell vacated is d, then a backward slide into d restores t. This is expressed by the following theorem:

Theorem 3.4 *(Sagan [4], page 114)*

$$j^d j_c(t) = t, \quad and \quad j_c j^d(t) = t.$$

Of course, we may want to make several slides in succession. This process is defined as follows.

Definition 3.9: A sequence of cells $(c_1, c_2, \cdots, c_l)$ is a slide sequence for a tableau t if we can form $t = t_0, t_1, \cdots, t_l$, where t_{c_k} is obtained from t_{i-1} by performing a slide into cell c_i. Two partial tableaux s and t are equivalent, written $s \cong t$, if t can be obtained from s by some sequence of slides.

The operation of jeu de taquin is a slide sequence which brings a tableau to normal shape, as in the following definition.

Definition 3.10: We perform jeu de taquin on a partial skew tableau, t, by performing an arbitrary slide sequence that brings the tableau to normal shape. The resultant normal tableau is denoted by $j(t)$.

Example 3.13 *Consider the skew tableau*

$$t = \begin{array}{ccc} & & 7 \\ & 6 & \\ 5 & & \end{array} .$$

We demonstrate the application of jeu de taquin to the tableau t by indicating with a dot the inner corner into which we perform a forward slide at each stage of the normalization process.

$$\begin{array}{ccc} & & 7 \\ \bullet & 6 & \\ 5 & & \end{array}, \quad \begin{array}{cc} \bullet & 7 \\ 5 & 6 \end{array}, \quad \begin{array}{ccc} \bullet & 6 & 7 \\ 5 & & \end{array}, \quad \begin{array}{ccc} 5 & 6 & 7 \end{array}.$$

The tableau which results from performing jeu de taquin, $j(t)$, is well defined, that is independent of the choice of slide sequence.

Theorem 3.5 *(Sagan [4], page 116.) Suppose that t is a partial skew tableau which is brought to a normal tableau t' by slides. Then t' is unique. Moreover, t' is the insertion tableau for π_t, the row word of $t(\pi)$.*

Equivalence of tableaux and Knuth equivalence are equivalent, as demonstrated by the following important theorem.

Theorem 3.6 *(Sagan [4], page 116) Let s and t be partial skew tableaux. Then*

$$s \cong t \Leftrightarrow s \overset{K}{\cong} t.$$

This is captured in the following important corollary.

Corollary 3.1. It follows from Lemma 3.1, Theorem 3.5 and Theorem 3.6, that two partial skew tableau, s and t, are Knuth equivalent, written as $s \overset{K}{\cong} t$, if their row words are Knuth equivalent as permutations. i.e. if $\pi_s \overset{K}{\cong} \pi_t$. These two conditions are equivalent.

Chapter 4

Symmetric Functions

The research by Hamel et al. [7], and McAven et al. [8] invokes the Littlewood-Richardson rule. In this chapter, we develop the theory of symmetric functions in sufficient detail to expound the Littlewood-Richardson rule.

This requires that we first define the ring of symmetric functions. We then introduce Schur functions. Next we define the characteristic map. Finally, we are in a position to state the Littlewood-Richardson rule.

In giving this coverage, we also introduce symmetric functions as a viable tool to conduct further research in this area. This limited coverage closely follows the book by Sagan [4]. For a more comprehensive coverage, the interested reader is referred to the book by MacDonald [17].

4.1 The Ring of Symmetric Functions.

The ring of symmetric functions is a set of power series invariant under the action of all the symmetric groups. Let $\boldsymbol{x} = \{x_1, x_2, x_3, \cdots\}$ be an infinite set of variables. Consider the formal power series ring $\mathbb{C}[[\boldsymbol{x}]]$. The monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l}$ has degree n if $n = \sum_i \lambda_i$. The power series $f(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$ is homogenous of degree n if every monomial in $f(\boldsymbol{x})$ has degree n.

For every n, there is a natural action of $\pi \in S_n$ on $f(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$, defined as

$$\pi f(x_1, x_2, x_3, \cdots) = f(x_{\pi 1}, x_{\pi 2}, x_{\pi 3}, \cdots),$$

where $\pi i = i$ for $i > n$.

Definition 4.1: Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ be a partition of n. The monomial symmetric function corresponding to λ is

$$m_\lambda = m_\lambda(\boldsymbol{x}) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l},$$

where the sum is over all distinct monomials having exponents $\lambda_1, \lambda_2, \cdots, \lambda_l$.

Example 4.1

$$m_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \cdots.$$

If $\lambda \vdash n$, then $m_\lambda(\boldsymbol{x})$ is homogeneous of degree n.

In the theory of symmetric functions, the number of variables is irrelevant, provided that there are sufficiently many variables to encode the representation that we are dealing with.

Definition 4.2: The ring of symmetric functions is

$$\wedge = \wedge(x) = \mathbb{C} m_\lambda,$$

that is, the vector space spanned by all the m_λ.

As $\wedge$ is closed under the product, it is really a ring, not just a vector space. There are some elements of $\mathbb{C}[[\boldsymbol{x}]]$ which are invariant under $\pi \in S_n$, but which are not in $\wedge$. For example, $\prod_{i\geq 1}(1+x_i)$ cannot be written as a finite linear combination of m_λ, so it is not in $\wedge$.

The decomposition of the ring of symmetric functions is

$$\wedge = \oplus_{n\geq 0} \wedge^n,$$

where $\wedge^n$ is the space spanned by all m_λ of degree n. This is a grading of $\wedge$ since $f \in \wedge^n$ and $g \in \wedge^m$ implies $fg \in \wedge^{n+m}$.

As the m_λ are independent, the space $\wedge^n$ has basis

$$\{m_\lambda \mid \lambda \vdash n\},$$

and therefore is of dimension $p(n)$, the number of partitions of n.

To construct other bases for $\wedge^n$, we introduce families of symmetric functions.

Definition 4.3: The n^{th} power sum symmetric function is

$$p_n = m(n) = \sum_{i\geq 1} x_i^n.$$

Definition 4.4: The n^{th} elementary symmetric function is

$$e_n = m_{(1^n)} = \sum_{i_i<\cdots<i_n} x_{i_1}\cdots x_{i_n}.$$

Definition 4.5: The n^{th} complete homogeneous symmetric function is

$$h_n = \sum_{\lambda\vdash n} m_\lambda = \sum_{i_i\leq\cdots\leq i_n} x_{i_1}\cdots x_{i_n}.$$

Example 4.2 *For $n=3$, we have*

$$\begin{aligned}
p_3 &= x_1^3+x_2^3+x_3^3+\cdots,\\
e_3 &= x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+\cdots,\\
h_3 &= x_1^3+x_2^3+\cdots+x_1^2x_2+\cdots+x_1x_2x_3+x_1x_2x_4+\cdots.
\end{aligned}$$

The elementary function e_n is just the sum of all square-free monomials of degree n. The complete homogeneous symmetric function is the sum of all monomials of degree n.

These functions are multiplicative. That is, for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$,

$$f_\lambda = f_{\lambda_1} f_{\lambda_2} \cdots f_{\lambda_l},$$

where each $f = p, e$ or h.

Example 4.3 *For* $\lambda = (2,1)$ *we have*

$$p_{(2,1)} = (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots).$$

It turns out that the power symmetric function, the elementary symmetric function and the homogeneous symmetric function all form bases for $\wedge^n$, as given by the following theorem.

Theorem 4.1 *(Sagan [4], page 154) The following are bases for* $\wedge^n$.

(1) $\{p_\lambda \mid \lambda \vdash n\}$.

(2) $\{e_\lambda \mid \lambda \vdash n\}$.

(3) $\{h_\lambda \mid \lambda \vdash n\}$.

Part (2) of this theorem says that every symmetric function is a polynomial in the elementary functions e_n.

4.2 Schur Functions

Another basis for $\wedge^n$ is the Schur functions. These functions are related to the irreducible representations of S_n and tableaux.

Definition 4.6: For any composition $\mu = (\mu_1, \mu_2, \cdots, \mu_l)$, there is a corresponding monomial weight in $\mathbb{C}[[\boldsymbol{x}]]$:

$$\boldsymbol{x}^{\mu} \stackrel{\text{def}}{=} x_1^{\mu_1} x_2^{\mu_2} \cdots x_l^{\mu_l}.$$

Definition 4.7: The weight of any generalized tableau T of shape λ is given by

$$\boldsymbol{x}^{T} \stackrel{\text{def}}{=} \prod_{(i,j)\in\lambda} x_{T_{i,j}} = \boldsymbol{x}^{\mu},$$

where μ is the content of T.

Example 4.4 *Given the tableau*

$$T = \begin{array}{ccc} 4 & 1 & 4 \\ 1 & 3 & \end{array},$$

then the weight of T is the monomial

$$\boldsymbol{x}^{T} = x_1^2 x_3 x_4^2.$$

Definition 4.8: Given a partition λ, the associated Schur function is

$$s_{\lambda}(\boldsymbol{x}) = \Sigma_T \boldsymbol{x}^{T},$$

where the sum is over all semi-standard λ-tableaux T.

Example 4.5 *For $\lambda = (2,1)$, some of the possible tableaux are*

$$\begin{matrix}1 & 1\\ 2 & \end{matrix},\quad \begin{matrix}1 & 2\\ 2 & \end{matrix},\quad \begin{matrix}1 & 1\\ 3 & \end{matrix},\quad \begin{matrix}1 & 3\\ 3 & \end{matrix},\quad \cdots \quad \begin{matrix}1 & 2\\ 3 & \end{matrix},\quad \begin{matrix}1 & 3\\ 2 & \end{matrix},\quad \begin{matrix}1 & 2\\ 4 & \end{matrix},\quad \begin{matrix}1 & 4\\ 2 & \end{matrix},$$

so that

$$s_{(2,1)}(\boldsymbol{x}) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + \cdots + 2x_1x_2x_3 + 2x_1x_2x_4 + \cdots.$$

If the partition $\lambda = (n)$, then a tableau with one row is a weakly increasing sequence of n positive integers. As this is a partition with n parts, it follows that

$$s_{(n)}(\boldsymbol{x}) = h_n(\boldsymbol{x}).$$

If the partition $\lambda = (1^n)$, then a tableau with one column is a strictly increasing sequence from top to bottom, giving

$$s_{(1^n)}(\boldsymbol{x}) = e_n(\boldsymbol{x}).$$

We next establish that the Schur functions form a basis for $\wedge^n$. To do this, we must first define an order on partitions.

Definition 4.9: Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$ be partitions of n. Then λ dominates μ, written $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \geq 1$. If $i > l$ (respectively, $i > m$), then we take λ_i (respectively, μ_i) to be zero.

Example 4.6 *When* $n = 6$, $(3,3) \trianglerighteq (2,2,1,1)$.

In order to show that the Schur functions form a basis for $\wedge^n$, we first establish that they are symmetric.

Theorem 4.2 *(Sagan [4], page 156) The function* $s_\lambda(\boldsymbol{x})$ *is symmetric.*

It follows that the Schur functions do form a basis for $\wedge^n$, as given by the following important theorem.

Theorem 4.3 *(Sagan [4], page 157) We have*

$$s_\lambda = \sum_{\mu \trianglelefteq x} K_{\lambda\mu} m_\mu,$$

where the sum is over partitions μ *(rather than compositions) and* $K_{\lambda\lambda} = 1$. *In this formulation,* $\triangleleft$ *denotes dominance ordering.*

Corollary 4.1 *The set* $\{s_\lambda \mid \lambda \vdash n\}$ *is a basis for* $\wedge^n$.

4.3 The Characteristic Map

In order to define the characteristic map, we must first define a class function.

Definition 4.10: A class function on a group G is a mapping, $f : G \to \mathbb{C}$, from G to the field of complex numbers such that $f(g) = f(h)$ whenever g and h are in the same conjugacy class of G. The set of all class functions on G is denoted by $R(G)$.

The sums and scalar multiples of class functions are again class functions, so $R(G)$ is a vector space over $\mathbb{C}$.

Let $R^n = R(S_n)$ be the space of class functions on S_n. The dimension of this space, $\dim R^n = \dim \wedge^n = p(n)$, is the number of partitions of n. Therefore, R^n and $\wedge^n$ are isomorphic as vector spaces. An inner product on R^n exists for which the irreducible characters on S_n form an orthonormal basis.

Definition 4.11: We also define an inner product on $\wedge^n$ as

$$< s_\lambda, s_\mu >= \delta_{\lambda\mu},$$

and sesquilinear extension, which is linear in the first variable and conjugate linear is the second variable.

Definition 4.12: A map exists which preserve these inner products. This is the characteristic map $ch^n : R^n \longrightarrow \wedge^n$, which is defined as

$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\mu p_\mu,$$

where χ_μ is the value of χ on the class μ.

In this definition z_μ is the order of the centralizer of a permutation of cycle type μ.

The characteristic map is unique in that it is the only map which preserves inner products. The characteristic map ch^n is linear. Applying it to the irreducible characters gives

$$ch^n(\chi^\lambda) = s_\lambda.$$

The characteristic map takes the space of class functions on S_n to $\wedge^n$, as given by the following theorem.

Theorem 4.4 *(Sagan [4], page 168) As the characteristic map ch^n takes one orthonormal basis to another, the map ch^n is an isometry between R^n and $\wedge^n$.*

The ring product of class functions $R = \oplus_n R^n$ is isomorphic to $\wedge = \oplus_n \wedge^n$ by the characteristic map $ch = \oplus_n ch^n$. The ring product $\wedge$ has the structure of a graded algebra. If χ and ψ are characters of S_n and S_m respectively, then to construct the corresponding product in R^n, we must find the character of S_{n+m}. The tensor product of $\chi \otimes \psi$ yields a character of $S_n \times S_m$.

Definition 4.13: We define a product on R by bilinear extension of

$$\chi . \psi = (\chi \otimes \psi) \uparrow^{S_{n+m}} .$$

Hence we have the following important theorem.

Theorem 4.5 *(Sagan [4], page 169) The map $ch : R \to \wedge$ is an isomorphism of algebras.*

This theorem is important as it establishes that the ring product of class functions is isomorphic to the ring of symmetric functions. Thus from the preceding discussion, this furnishes an approach to representation of S_{n+m} using symmetric functions and group character theory.

4.4 The Littlewood-Richardson Rule

Young's rule, Theorem 2.2, states that

$$M^{\mu} \cong \oplus_{\lambda} K_{\lambda\mu} S^{\lambda},$$

where $K_{\lambda\mu}$ is the number of semi-standard tableaux of shape λ and content μ. In this formulation S^{λ} is the Specht module associated with the partition λ, and M^{μ} is the permutation module corresponding to μ. This result can be viewed in terms of characters or symmetric functions.

If $\mu \vdash n$, then M^{μ} is a module for the induced character $1_{S_{\mu}} \uparrow^{S_n}$. The definitions of the character and the tensor product give that

$$1_{S_{\mu}} = 1_{S_{\mu_1}} \otimes 1_{S_{\mu_2}} \otimes \cdots \otimes 1_{S_{\mu_m}},$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$. This may be written as

$$1_{S_{\mu_1}} . 1_{S_{\mu_2}} \cdots 1_{S_{\mu_m}} = \sum_{\lambda} K_{\lambda\mu} \chi^{\lambda}.$$

Applying the characteristic map gives

$$s_{(\mu_1)}s_{(\mu_2)}\cdots s_{(\mu_m)} = \sum_{\lambda} K_{\lambda\mu} s_{\lambda}.$$

Example 4.7

$$M^{(3,2)} = S^{(3,2)} + S^{(4,1)} + S^{(5)},$$

with the corresponding tableaux

$$\begin{array}{ccc} 1\ 1\ 1 & 1\ 1\ 1\ 2 & 1\ 1\ 1\ 2\ 2 \\ 2\ 2 & 2 & \end{array},\ ,\ ,$$

which may be written as

$$1_{S_3}.1_{S_2} = \chi^{(3,2)} + \chi^{(4,1)} + \chi^{(5)},$$

or

$$s_{(3)}s_{(2)} = s_{(3,2)} + s_{(4,1)} + s_{(5)}.$$

The computation of the expansion

$$s_{\mu}s_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu} s_{\lambda},$$

on arbitrary partitions μ and ν is equivalent to the computation of the irreducibles in

$$\chi^{\mu}\chi^{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\chi^{\lambda},$$

or

$$(S^{\mu} \otimes S^{\nu}) \uparrow = \oplus_{\lambda} c^{\lambda}_{\mu\nu} S^{\lambda},$$

where $|\mu| + |\nu| = n$, and the $c^{\lambda}_{\mu\nu}$ are called the Littlewood-Richardson coefficients.

The Littlewood-Richardson rule gives a combinatorial interpretation of these coefficients. These coefficients also arise in the expansion of the skew Schur functions, which are the Schur functions associated with skew shapes. These are the subject of the following theorem.

Theorem 4.6 *(Sagan [4], page 175) If we define* $s_{\lambda}(\boldsymbol{x},\boldsymbol{y}) = s_{\lambda}(x_1, x_2, \cdots, y_1, y_2, \cdots)$*, then*

$$s_{\lambda}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\mu \subseteq \lambda} s_{\mu}(\boldsymbol{x}) s_{\lambda/\mu}(\boldsymbol{y}).$$

The skew symmetric function $s_{\lambda/\mu}(y)$ can be expressed as a linear combination of ordinary (non-skew) Schur functions. This is associated with the Littlewood-Richardson coefficients, as given by the following theorem.

Theorem 4.7 *(Sagan [4], page 175) If the* $c^{\lambda}_{\mu\nu}$ *are Littlewood-Richardson coefficients where* $|\mu| + |\nu| = |\lambda|$*, then*

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

One more definition is required to state the Littlewood-Richardson rule.

Definition 4.14: A ballot sequence or lattice permutation is a sequence of positive integers $\pi = i_1 i_2 \cdots i_n$ such that

$$\pi_k = i_1 i_2 \cdots i_k \quad \text{for all} \quad 1 \leq k \leq n,$$

and for any positive integer l, the number of l's in π_k is at least as large as the number of $(l+1)$'s in that prefix.

A reverse ballot sequence or reverse lattice permutation is a permutation π for which π^r is a ballot sequence. In this definition, π^r is the reversal of the permutation π in one-row format.

Example 4.8 *As an example, the sequence*

$$\pi = 1 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \;\; 1 \;\; 3,$$

is a lattice permutation, but

$$\sigma = 1 \;\; 2 \;\; 3 \;\; 2 \;\; 1 \;\; 1 \;\; 3,$$

is not, because the leading term 1 2 3 2 has more 2's than 1's.

Lattice permutations provide an encoding of standard tableaux. Given a standard tableau t with n elements the sequence $\pi = i_1 i_2 \cdots i_n$ is then formed where $i_k = i$ if k appears in row i of t. Entries less than or equal to k form a partition with weakly decreasing parts. This ensures the ballot sequence condition on π_k.

Example 4.9 *The preceding lattice permutation* π *encodes the tableau*

$$\begin{array}{ccc} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & 7 & \end{array} .$$

We are now in a position to be able to state the Littlewood-Richardson Rule.

Theorem 4.8 ***Littlewood-Richardson Rule (Sagan [4], page 177)***

The value of each Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of semistandard tableaux t such that

(1) *t has shape λ/μ and content ν,*

(2) *the row word of t, π_t, is a reverse lattice permutation.*

A bijection exists between semi-standard tableaux t of shape λ/μ and semi-standard tableaux of normal shape, that is,

$$t \xrightarrow{j} U,$$

where U is the set of semi-standard tableaux of normal shape. This map j is just the jeu de taquin.

Theorem 4.9 *(Sagan [4], page 178.) If a tableau T' can be obtained from a tableau T by a sequence of slides, then π_T is a reverse lattice permutation if and only if $\pi_{T'}$ is also a reverse lattice permutation.*

Example 4.10 *Given the product $s_{(2,1)}s_{(2,2)}$, then the tableaux corresponding to lattice permutations with content $(2,2)$ and shape $\lambda/(2,1)$ for some λ are:*

$$
\begin{array}{llll} \bullet & \bullet & 1 & 1 \\ \bullet & 2 & 2 & \\ & & & \\ & & & \end{array},
\quad
\begin{array}{llll} \bullet & \bullet & 1 & 1 \\ \bullet & 2 & & \\ 2 & & & \\ & & & \end{array},
\quad
\begin{array}{lll} \bullet & \bullet & 1 \\ \bullet & 1 & 2 \\ 2 & & \\ & & \end{array},
\quad
\begin{array}{lll} \bullet & \bullet & 1 \\ \bullet & 1 & \\ 2 & 2 & \\ & & \end{array},
\quad
\begin{array}{lll} \bullet & \bullet & 1 \\ \bullet & 2 & \\ 1 & & \\ 2 & & \end{array},
\quad
\begin{array}{ll} \bullet & \bullet \\ \bullet & 1 \\ 1 & 2 \\ 2 & \end{array}.
$$

This gives

$$s_{(2,1)}s_{(2,2)} = s_{(4,3)} + s_{(4,2,1)} + s_{(3^2,1)} + s_{(3,2^2)} + s_{(3,2,1^2)} + s_{(2^3,1)}.$$

Example 4.11 *Given the outer shape $s_{(5,3,2,1)}$, then we wish to find the coefficients of this shape in $s_{(3,2,1)}s_{(3,2)}$. The tableaux are*

$$
\begin{array}{lllll} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & \bullet & 1 & & \\ \bullet & 2 & & & \\ 2 & & & & \end{array},
\quad
\begin{array}{lllll} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & \bullet & 2 & & \\ \bullet & 1 & & & \\ 2 & & & & \end{array},
\quad
\begin{array}{lllll} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & \bullet & 2 & & \\ \bullet & 2 & & & \\ 1 & & & & \end{array}.
$$

This gives that

$$c^{(5,3,2,1)}_{(3,2,1)(3,2)} = 3.$$

Chapter 5

An Introduction to the Transition Matrix between Symmetric Group Bases

We consider here representations of the symmetric group on n elements, written as S_n. The number of conjugacy classes of the symmetric group is the number of inequivalent irreducible representations of the symmetric group. Each conjugacy class is associated with a partition of n. Thus the irreducible representations of S_n may be labeled by partitions of n. Associated with each partition, λ, of n is a Young tableaux of shape λ. The number of Young tableaux for a given partition of λ, of n, is equal to the degree of their reducible representation of S_n associated with the partition, λ, of n.

We will be considering the representation of the symmetric group in several different bases.

5.1 Adaption to a Basis

The number of conjugacy classes of the symmetric group in equal to the number of inequivalent irreducible representations. Partitions of n can be used to label the classes of S_n, so they can also be used to label the irreducible representations of S_n.

Having identified the irreducible representations of S_n, we consider the basis vectors which span the irreducible representations. To do this, we consider the behavior of the basis vectors under the chain of subgroups

$$S_n \supset S_{n-1} \supset S_{n-2} ... \supset S_2.$$

Now consider the reduction of the irreducible representations from S_n to S_{n-1}. The irreducible representations of S_{n-1} may also be labeled by partitions. The partitions of S_n and S_{n-1} differ in only one part, smaller by one in the S_{n-1} partition. This is equivalent to the Ferrers diagram on $(n-1)$ nodes obtained by removing an outer corner from the Ferrers diagram for the S_n irreducible representation. Recording the S_n irreducible representation and the S_{n-1} irreducible representation gives a basis for the irreducible representations in the $S_n \supset S_{n-1}$ subgroup chain.

This process does not uniquely label the basis vectors, since the S_{n-1} irreducible representations will usually be of dimension greater than one. So we need to extend the subgroup chain as in equation 5.1 to obtain a set of irreducible representations which do uniquely label each basis vector. This is illustrated in Figure 5.1.

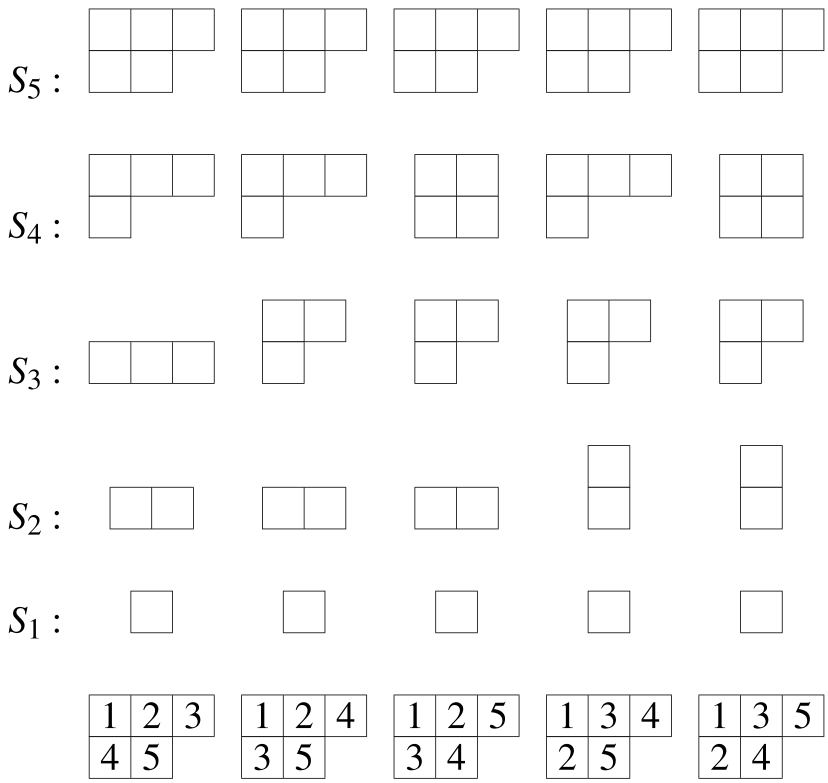


Figure 5.1: The set of irreducible representations (partitions) associated with each of the basis vectors of the irreducible [32].

The sequence of diagrams is an extended way of labeling the basis vectors. Since the Ferrers diagram in the sequence differs in only one node between steps, we can associate with each sequence a numbered Ferrers diagram. This is the Young tableau introduced in Chapter 2. The number of Young tableaux is equal to the dimension of the irreducible representation. At the bottom of Figure 5.1, we give the Young tableaux for the basis vectors labelled by the sequences shown.

It is possible to adapt the basis to other systems than the chain of subgroups. In general, the differently adapted bases are called non-standard bases. In particular, we are interested in split bases such as direct product adapted bases. These bases are adapted to direct products of subgroups of the form

$$S_{n_1} \times S_{n_2} \times ... \times S_{n_k}.$$

One can label the basis vectors of the split basis by tuples of tableaux obtained by removing the first n_k nodes from the Young tableau, then n_{k-1} nodes and so on. These tableaux determine the representation matrices of the adjacent transpositions in the split bases using the method described above for the standard basis tableaux. The first tableau is used if the adjacent transposition is in S_{n_1}, the second tableau is used if the adjacent transposition is in S_{n_2}, and so on. The transpositions between the factor subgroups, the bridging transpositions cannot be calculated in this manner. This is the subject of ongoing research.

This requires an ordering on tableaux and tuples of tableaux. This is discussed in the next section.

5.2 Orderings on Tableaux

First, we wish to impose an order on tableaux of the same shape. To do this, we define a total order on tableaux known as first letter order or dictionary order, as follows.

Definition 5.1: ([5], page 36.) A list of tableaux my be placed in first letter order or dictionary order by assuming that tableau t_1 precedes t_2 if the integers in the first s rows of each tableau are the same and the the first s integers in the $(r+1)^{th}$ row are the same but the $(s+1)^{th}$ integer in the $(r+1)^{th}$ row of t_1 precedes the $(s+1)^{th}$ integer in the $(r+1)^{th}$ row of t_2.

Example 5.1 *Let s and t be the tableaux,*

$$s = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & & & \end{matrix} ,$$

and

$$t = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 8 & \\ 7 & & & \end{matrix} .$$

Then tableau s precedes tableau t in first letter order.

Example 5.2 *Let $\lambda = (3,2,1)$. There are sixteen standard tableaux of shape λ, ie.*

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

$t_a \quad t_b \quad t_c \quad t_d$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

$t_e \quad t_f \quad t_g \quad t_h$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

$t_i \quad t_j \quad t_k \quad t_l$

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&5\\\cline{1-2} 4\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

$t_m \quad t_n \quad t_o \quad t_p$

These sixteen tableaux are shown in first letter order.

Intuitively this is the order in which the tableau elements would appear in a dictionary. In defining matrix representations, last letter order is sometimes used. This is introduced later in Definition 5.3. Second, we wish to impose an order on tableaux which have different shape. To do this, we define partition order as follows.

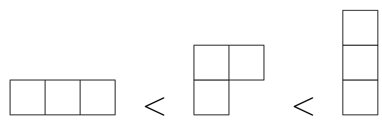
Definition 5.2: ([8], page 3.) Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_e)$ and $\mu = (\mu_1, \mu_2, ..., \mu_m)$ be partitions of n. Then partition λ precedes partition μ in partition order, written $\lambda < \mu$, if, for some order index i,

$$\lambda_j = \mu_j \ for \ j < i \ and \ \lambda_i = \mu_i.$$

Example 5.3 *The partitions of $n = 3$, we have that*

$$(3) < (2,1) < (1^3).$$

This is illustrated by the tableau shapes



Intuitively tableau s precedes tableau t in partition order if tableau s is short and fat and tableau t is long and thin.

We wish to order tuples of tableaux which have been normalised using jeu de taquin. To do this, we apply a combination of partition order and first letter order, as follows.

Algorithm PF ([8], page 4.)

Step PF-1 To order tuples of tableaux, we first apply partition order to the first tableaux in the tuples if the shapes of the first tableaux are different.

PF-2 Apply first letter order to the first tableaux in tuples if the shapes of the first tableaux are identical.

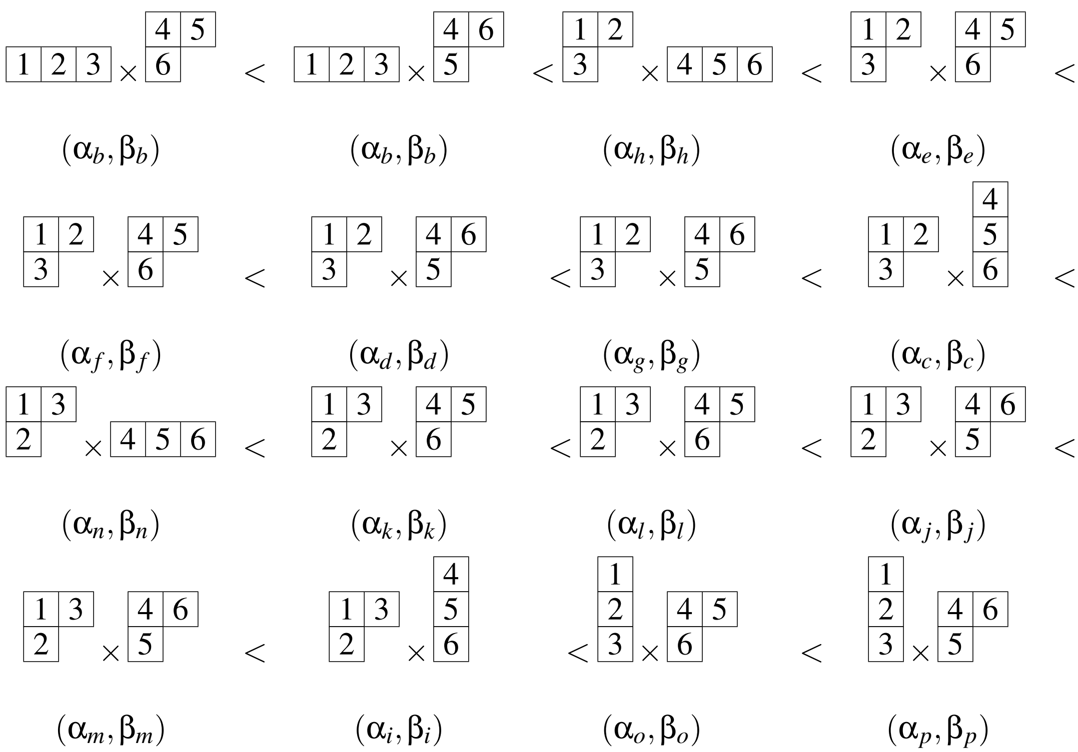
PF-3 Apply partition order to the second tableaux in the tuples if the first tableaux are identical and the shapes of the second tableaux are different.

PF-4 Apply first letter order to the second tableaux in the tuples if the first tableaux are identical.

PF-5 If the first and second tableaux in the tuples are identical, apply partition order and first letter order as above to the third tableaux, and so on, until the tuples are arranged in order.

PF-6 If multiplicities occur, order the tuples according to first letter order of the standard tableaux which mapped to the tuple.

Example 5.4 *Let* $\lambda = (3,2,1)$ *be a partition of* 6. *We consider again the removal of three nodes from the sixteen standard tableaux of shape* λ*, corresponding to the direct product subgroup* $S_3 \times S_3$. *The tableau pairs are shown arranged in order according to this algorithm.*



We are interested in the representation matrices corresponding to direct product subgroups of the form

$$S_{n_1} \times S_{n_2} \times \ldots \times S_{n_k}$$

where $n_1 + n_2 + \ldots + n_k = n$.

In terms of tableaux, this corresponds to the k-tuple of tableaux obtained by first removing n_k nodes from the tableau for S_n, then n_{k-1} nodes, and so on, until there are n_1 nodes left. Each tableau in the k-tuple is normalised using jeu de taquin.

Note that this correspondence is many-to-one, so that a single k-tuple of tableaux may have several standard tableaux which map to it.

Example 5.5 *Let $\lambda = (3,2,1)$ be a partition of 6. We are interested in the direct product subgroup $S_3 \times S_3$. This corresponds to the removal of three nodes from the sixteen standard tableaux of shape λ. For the sake of simplicity, we choose to remove the nodes containing the integers 4, 5 and 6, though we could choose the integers for removal arbitrarily.*

We show the tableaux pairs obtained by removal of the nodes.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \quad \Rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \times \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 6 \\ \cline{1-1}\end{array}$$

$$t_a \qquad\qquad (\alpha_a, \beta_a)$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \quad \Rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \times \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 5 \\ \cline{1-1}\end{array}$$

$$t_b \qquad\qquad (\alpha_b, \beta_b)$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|}\hline 4\\ \hline 5\\ \hline 6\\ \hline\end{array}$$

t_c $\qquad$ (α_c, β_c)

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 4&6\\ \hline 5 \\ \cline{1-1}\end{array}$$

t_d $\qquad$ (α_d, β_d)

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 4&5\\ \hline 6 \\ \cline{1-1}\end{array}$$

t_e $\qquad$ (α_e, β_e)

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 4&5\\ \hline 6 \\ \cline{1-1}\end{array}$$

t_f $\qquad$ (α_f, β_f)

$$\begin{array}{|c|c|c|}\hline 1&2&6\\ \hline 3&4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 4&6\\ \hline 5 \\ \cline{1-1}\end{array}$$

t_g $\qquad$ (α_g, β_g)

$$\begin{array}{|c|c|c|}\hline 1&2&6\\ \hline 3&5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|c|}\hline 4&5&6\\ \hline\end{array}$$

t_h $\qquad$ (α_h, β_h)

$$\begin{array}{ccc} 1 & 3 & 4 \\ 3 & 5 & \\ 6 & & \end{array} \Rightarrow \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \times \begin{array}{c} 4 \\ 5 \\ 6 \end{array}$$

$$t_i \qquad (\alpha_i, \beta_i)$$

$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{array} \Rightarrow \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \times \begin{array}{cc} 4 & 6 \\ 5 & \end{array}$$

$$t_j \qquad (\alpha_j, \beta_j)$$

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & & \end{array} \Rightarrow \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \times \begin{array}{cc} 4 & 5 \\ 6 & \end{array}$$

$$t_k \qquad (\alpha_k, \beta_k)$$

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array} \Rightarrow \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \times \begin{array}{cc} 4 & 5 \\ 6 & \end{array}$$

$$t_l \qquad (\alpha_l, \beta_l)$$

$$\begin{array}{ccc} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{array} \Rightarrow \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \times \begin{array}{cc} 4 & 6 \\ 5 & \end{array}$$

$$t_m \qquad (\alpha_m, \beta_m)$$

$$\begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \Rightarrow \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \times \begin{array}{ccc} 4 & 5 & 6 \end{array}$$

$$t_n \qquad (\alpha_n, \beta_n)$$

$$\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1}\end{array} \Rightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \times \begin{array}{|c|c|}\hline 4&5\\\hline 6\\\cline{1-1}\end{array}$$

$$t_o \qquad (\alpha_o, \beta_o)$$

$$\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array} \Rightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \times \begin{array}{|c|c|}\hline 4&6\\\hline 5\\\cline{1-1}\end{array}$$

$$t_p \qquad (\alpha_p, \beta_p)$$

Note that in this example, the standard tableaux are in first letter order but the derived tableaux pairs are not. To impose an order on the tableaux pairs, we apply Algorithm PF. In this example, we also see instances of the many-to-one correspondence between Young tableaux of shape λ and tableaux pairs, (α, β). For example

1. Both tableaux, t_d and t_g, map to the tableau pair $(\alpha_d, \beta_d) = (\alpha_g, \beta_g)$;

2. Both tableaux, t_e and t_f, map to the tableau pair $(\alpha_e, \beta_e) = (\alpha_f, \beta_f)$;

3. Both tableaux, t_j and t_m, map to the tableau pair $(\alpha_j, \beta_j) = (\alpha_m, \beta_m)$;

4. Both tableaux, t_k and t_e, map to the tableau pair $(\alpha_k, \beta_k) = (\alpha_e, \beta_e)$.

These are examples of product multiplicity in the $S_n - S_{a,b}$ basis. Product multiplicity is resolved by using Algorithm PF.

We wish to order tableaux according to last letter order, as follows.

Definition 5.3: ([8], page 3.) An integer is said to be lower in a tableau if it occurs either in a lower row or in the same row and to be the right of a second integer. Let λ be a partition. The set of tableaux of shape λ may be placed in last letter order by placing tableaux in which in largest letter occurs lower in the tableau later in the ordering. If the largest letter is in the same partition, tableaux are ordered according to the second largest letter, and so on.

Example 5.6 *Let $\lambda = (3,2)$. The last-lettering ordering of Young tableaux of shape λ gives the tableau sequence.*

$$\begin{array}{ccc}1&3&5\\2&4&\end{array} < \begin{array}{ccc}1&2&5\\3&4&\end{array} < \begin{array}{ccc}1&3&4\\2&5&\end{array} < \begin{array}{ccc}1&2&4\\3&5&\end{array} < \begin{array}{ccc}1&2&3\\4&5&\end{array}.$$

Example 5.7 *Let $\lambda = (3,2,1)$. The sixteen standard tableaux of shape λ are shown arranged in last letter order below.*

$$\begin{array}{ccc}1&2&3\\4&5&\\6&&\end{array} < \begin{array}{ccc}1&2&4\\3&5&\\6&&\end{array} < \begin{array}{ccc}1&3&4\\2&5&\\6&&\end{array} < \begin{array}{ccc}1&2&5\\3&4&\\6&&\end{array} <$$

$$\begin{array}{ccc}1&3&5\\2&4&\\6&&\end{array} < \begin{array}{ccc}1&2&3\\4&6&\\5&&\end{array} < \begin{array}{ccc}1&2&4\\3&6&\\5&&\end{array} < \begin{array}{ccc}1&3&4\\2&6&\\5&&\end{array} <$$

$$\begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array} < \begin{array}{ccc}1&3&5\\2&6&\\4&&\end{array} < \begin{array}{ccc}1&4&5\\2&6&\\3&&\end{array} < \begin{array}{ccc}1&2&6\\3&4&\\5&&\end{array} <$$

$$\begin{array}{ccc}1&3&6\\2&4&\\5&&\end{array} < \begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array} < \begin{array}{ccc}1&3&6\\2&5&\\4&&\end{array} < \begin{array}{ccc}1&4&6\\2&5&\\3&&\end{array}.$$

5.3 The Young-Yamanouchi Representation

In the literature, the most common representation of the symmetric group is the Young-Yamanouchi representation. The derivation of this representation is due to Yamanouchi and is given in Hamermesh [12], pages 214-223.

The representation matrices associated with the Young-Yamanouchi (or YY) basis vectors are defined in terms of the axial distance between two nodes in a tableau. This definition is as follows:

Definition 5.3: ([7], page 3.) Let i be the node in row r_i and column c_i of a tableau. Let j be the node in row r_j and column c_j of the same tableau. The axial distance from node i to node j is

$$\tau_{ij} = (c_j - r_j) - (c_i - r_i).$$

The reciprocal of the axial distance, ρ, is defined as

$$\rho = \frac{1}{\tau}.$$

The representation matrix in the YY basis is also defined in terms of the Young-Yamanouchi, on YY, symbol. This symbol is computed by the following algorithm.

Algorithm YY ([7], page 3.)

Step YY-1 Set $x = n$.

Step YY-2 Locate the symbol labelled x in the Young tableau. Write down the row of the tableau in which this symbol appears.

Step YY-3 Remove x from the Young tableau.

Step YY-4 Set $x = x - 1$.

Step YY-5 If $x = 0$, the algorithm terminates. Otherwise, go to Step Step YY-2.

Example 5.8 *Given the Young tableaux*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\quad ,$$

then the Young-Yamanouchi symbol computed by the above algorithm is

$$3\ 2\ 2\ 1\ 1\ 1.$$

We are now in a position to define the Young-Yamanouchi representation matrix. Note that since the symmetric group is generated by adjacent transpositions, it suffices to compute the Young-Yamanouchi representation matrices for the adjacent transposition of the form $(k-1,k)$. These matrices are written as $M^{\lambda}_{(k-1,k)}$.

Definition 5.4: ([7], page 3.) First, order the tableaux for the partition, λ, according to last letter order. Also order the Young-Yamanouchi symbols by last letter order. Then the Young-Yamanouchi representation matrix, $M^{\lambda}_{(k-1,k)}$, for the transposition $(k-1,k)$ is defined as follows :

(1) The matrix, $M^{\lambda}_{(k-1,k)}$, has $+1$ is position (r,r) if, in the r^{th} YY symbol, the $(n-k+1)^{\text{th}}$ and the $(n-k)^{\text{th}}$ elements are identical, that is, if the r^{th} tableau has $(k-1)$ and k in the same row.

(2) The matrix, $M^{\lambda}_{(k-1,k)}$ has -1 in position (r,r) if, in the r^{th} YY symbol, the $(n-k+1)^{\text{th}}$ element, α, is one more than the $(n-k)^{\text{th}}$ element, β, and there does not exist another YY symbol which is identical to this YY symbol except that its $(n-k+1)^{\text{th}}$ element is β and its $(n-k)^{\text{th}}$ element is α, that is, the r^{th} tableau has $(k-1)$ and k in the same column.

(3) The matrix, $M^{\lambda}_{(k-1,k)}$ has $-\rho$ in position (r,r), $\sqrt{1-\rho^2}$ in positions (r,s) and (s,r) and ρ in position (s,s) if $r<s$ and the r^{th} and s^{th} YY symbols are identical except that the $(n-k+1)^{\text{th}}$ element of the r^{th} YY symbol is the $(n-k)^{\text{th}}$ element of the s^{th} YY symbol and vice-versa, that is, the s^{th} tableau is obtained from the r^{th} tableau by interchanging $(k-1)$ and k.

(4) The matrix, $M^{\lambda}_{(k-1,k)}$, has zero in all other positions.

Example 5.9 *The YY representation matrices for the transpositions* (12), (23), (34), (45) *and* (56) *are shown below.*

$$[(12)] = \begin{vmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& -1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& -1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& -1 &&&&&&&& \\
&&&&&&&& 1 &&&&&&& \\
&&&&&&&&& -1 &&&&&& \\
&&&&&&&&&& -1 &&&&& \\
&&&&&&&&&&& 1 &&&& \\
&&&&&&&&&&&& -1 &&& \\
&&&&&&&&&&&&& 1 && \\
&&&&&&&&&&&&&& -1 & \\
&&&&&&&&&&&&&&& -1
\end{vmatrix}$$

$$
[(2,3)] = \begin{vmatrix}
1 \\
& \frac{-1}{2} & \sqrt{\frac{3}{2}} \\
& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&& \frac{-1}{2} & \sqrt{\frac{3}{2}} \\
&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&& 1 \\
&&&&&& \frac{-1}{2} & \sqrt{\frac{3}{2}} \\
&&&&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&&&&& \frac{-1}{2} & \sqrt{\frac{3}{2}} \\
&&&&&&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&&&&&&& 1 \\
&&&&&&&&&&& \frac{-1}{2} & \sqrt{\frac{3}{2}} \\
&&&&&&&&&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&&&&&&&&&& \frac{-1}{2} & \sqrt{\frac{3}{2}} \\
&&&&&&&&&&&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&&&&&&&&&&&& 1
\end{vmatrix}
$$

$$
[(3,4)] = \begin{vmatrix}
\frac{-1}{3} & \sqrt{\frac{8}{3}} \\
\sqrt{\frac{8}{3}} & \frac{1}{3} \\
&& 1 \\
&&& 1 \\
&&&& 1 \\
&&&&& \frac{-1}{3} & \sqrt{\frac{8}{3}} \\
&&&&& \sqrt{\frac{8}{3}} & \frac{1}{3} \\
&&&&&&& 1 \\
&&&&&&&& 1 \\
&&&&&&&&& \frac{-1}{3} & \sqrt{\frac{8}{3}} \\
&&&&&&&&& \sqrt{\frac{8}{3}} & \frac{1}{3} \\
&&&&&&&&&&& 1 \\
&&&&&&&&&&&& 1 \\
&&&&&&&&&&&&& 1 \\
&&&&&&&&&&&&&& \frac{-1}{3} & \sqrt{\frac{8}{3}} \\
&&&&&&&&&&&&&& \sqrt{\frac{8}{3}} & \frac{1}{3}
\end{vmatrix}
$$

$$
[(4,5)] = \begin{vmatrix}
1 \\
& \frac{-1}{2} && \sqrt{\frac{3}{2}} \\
&& \frac{-1}{2} \\
& \sqrt{\frac{3}{2}} && \frac{1}{2} \\
&& \sqrt{\frac{3}{2}} && \frac{1}{2} \\
&&&&& -1 \\
&&&&&& \frac{-1}{4} && \sqrt{\frac{15}{4}} \\
&&&&&&& \frac{-1}{4} && \sqrt{\frac{15}{4}} \\
&&&&&& \sqrt{\frac{15}{4}} && \frac{1}{4} \\
&&&&&&& \sqrt{\frac{15}{4}} && \frac{1}{4} \\
&&&&&&&&&& 1 && \sqrt{\frac{3}{2}} \\
&&&&&&&&&&& \frac{-1}{2} && \sqrt{\frac{3}{2}} \\
&&&&&&&&&& \sqrt{\frac{3}{2}} && \frac{1}{2} \\
&&&&&&&&&&& \sqrt{\frac{3}{2}} && \frac{1}{2} \\
&&&&&&&&&&&&&& -1
\end{vmatrix}
$$

$$
[(5,6)] = \left|\begin{array}{cccccccccccccccc}
\frac{-1}{2} & & & & & \sqrt{\frac{3}{2}} & & & & & & & & & & \\
& \frac{-1}{2} & & & & & \sqrt{\frac{3}{2}} & & & & & & & & & \\
& & \frac{-1}{2} & & & & & \sqrt{\frac{3}{2}} & & & & & & & & \\
& & & \frac{-1}{4} & & & & & & & & \sqrt{\frac{15}{4}} & & & & \\
& & & & \frac{-1}{4} & & & & & & & & \sqrt{\frac{15}{4}} & & & \\
\sqrt{\frac{3}{2}} & & & & & \frac{1}{2} & & & & & & & & & & \\
& \sqrt{\frac{3}{2}} & & & & & \frac{1}{2} & & & & & & & & & \\
& & \sqrt{\frac{3}{2}} & & & & & \frac{1}{2} & & & & & & & & \\
& & & & & & & & \frac{-1}{2} & & & & & & & \\
& & & & & & & & & \frac{-1}{2} & & & & & & \\
& & & & & & & & & & \frac{-1}{2} & & & & & \\
& & & \sqrt{\frac{15}{4}} & & & & & & & & \frac{1}{4} & & & & \\
& & & & \sqrt{\frac{15}{4}} & & & & & & & & \frac{1}{4} & & & \\
& & & & & & & & \sqrt{\frac{3}{2}} & & & & & \frac{1}{2} & & \\
& & & & & & & & & \sqrt{\frac{3}{2}} & & & & & \frac{1}{2} & \\
& & & & & & & & & & \sqrt{\frac{3}{2}} & & & & & \frac{1}{2}
\end{array}\right|
$$

The derivation of the representation matrix is due to Yamanouchi and is given in Hamermesh [12], pages 214-223.

5.4 The Transition Matrix

It is sufficient to derive the representation matrix for adjacent transpositions, since the symmetric group is generated by adjacent transpositions, as described on page 68.

An important basis we will be considering is the split basis. This is the representation of the symmetric group adapted to a direct product subgroup $S_a \times S_b$, where $a+b=n$. In this representation, adjacent transposition will be represented by the Young-Yamanouchi matrix provided that it falls in the subgroup S_a on the subgroup S_b. There is one transposition $(a, a+1)$ whose elements fall in S_a and S_b. The representation of this element in the split basis is the subject of ongoing research. Indeed, this is the crux of the research problem addressed in this thesis.

More generally, we are interested in a representation of the symmetric group adapted to a direct product of subgroups of the form

$$S_{n_1} \times S_{n_2} \times ... \times S_{n_k}$$

where

$$\sum_i n_i = n.$$

In his case there will be $k-1$ bridging transpositions.

Consider another such direct product of subgroup of the form

$$S_{m_1} \times S_{m_2} \times ... \times S_{m_l}$$

where

$$\sum_i m_i = n.$$

We are again interested in a representation adapted to this direct product.

In this research, we are interested in the matrix which transforms between these two bases. This is known as the transition matrix. It allows coefficients in the representation matrices, for example, $S_a \times S_b$, to be related to coefficients in the representation matrix for $S_c \times S_d$. The representation for $S_a \times S_b$ will consist of blocks for each of the generators, as will the representation matrices for $S_c \times S_d$. The transition matrix will relate the coefficients of each block between the two sets of representation matrices.

Chapter 6

The Skew Representation

The research of Hamel et al. [7], and McAven et al. [8], (see Chapter 7) investigates the removal of nodes from a tableau and the corresponding representation. They do this by normalizing the resulting skew tableaux and applying the Young-Yamanouchi basis to the normalized tableaux.

We wish to consider some possible means of furthering this research. One possible method is introduced in this chapter. It involves the concept of a skew representation. In this representation, skew tableau are not normalized, but a representation is applied directly to the skew tableau. This representation is an extension of the Young-Yamanouchi basis of Chapter 5, defined in Section 5.3. To make this definition, we must first introduce Young's Raising Operator and lattice permutations.

This limited coverage closely follows the treatment given in the book by Robinson [5]. For a more comprehensive treatment, the interested reader is thus referred to the book by

Robinson [5].

6.1 Young's Raising Operator R_{ik}

Instead of writing the rows of the tableau t above one another as we usually do, we may write them disjointly, giving a special case of a skew diagram, viz

$$
\begin{array}{llll}
\cdot\ \cdot\ \cdot\ \cdots\ \lambda_1 \text{ nodes} & & & \\
 & \cdot\ \cdot\ \cdot\ \cdots\ \lambda_2 \text{ nodes} & & \\
 & & \vdots & \\
 & & & \cdot\ \cdot\ \cdot\ \cdots\ \lambda_h \text{ nodes}
\end{array}
$$

We designate this skew diagram by a direct product of component tableaux

$$\lambda_1 \times \lambda_2 \times \cdots \times \lambda_h.$$

The arrangement of the disjoint rows and the order of the factors is immaterial. The number of such standard skew tableaux is

$$f_\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_h!}.$$

This is the degree of the permutation representation of S_n, M^λ, induced by the identity representation of the subgroup

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_h}.$$

The matrices of the transpositions $(r, r+1)$ are computed as described in Chapter 5 if we take $\rho = 0$.

In order to define skew representations and Young's Raising Operator, we need to define dictionary order of tableaux.

Definition 6.1: Lists of tableaux may be placed in dictionary order by assuming that tableau t precedes tableau t', written $t < t'$, if the integers in the first r rows of each tableau are the same, and the first s integers in the $(r+1)+h$ row are the same, but the $(s+1)+h$ integer in the $(r+1)+h$ row of t precedes the $(s+1)+h$ integer in the $(r+1)+h$ integer in the $(r+1)+h$ row of t'.

Example 6.1 *Let t and t' be the tableaux*

$$t = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & & & \end{matrix} \qquad \textit{and} \qquad t' = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 8 & \\ 7 & & & \end{matrix} \quad .$$

Then tableau t precedes tableau t' in dictionary order.

The operator theory of representations is based on Young's substitutional equation:

$$\frac{\sum' S_\lambda}{\lambda_1!\lambda_2!\cdots\lambda_k!} = \sum(\prod R_{ik})\frac{n!}{f^\lambda}t^\lambda,$$

where $\sum'$ indicates summation over all $n!$ arrangements of the symbols of the skew diagram above. For $i<k$, the operator R_{ik} represents the raising of a node from the k^{th} row to the i^{th} row of t to yield a new skew diagram s. The compound operator $\sqcap R_{ik}$ represents successive raisings of nodes in t where $i = 1,2,\cdots,h-1$; $k = 2,3,\cdots,h$. The s tableau thus obtained precede the tableau t in dictionary order.

The substitutional equation above gives the reduction of the permutation representation as

$$\begin{aligned} [\lambda_1]\otimes[\lambda_2]\otimes\cdots\otimes[\lambda_h] &= \sum(\prod R_{ik})[\lambda] \\ &= [\lambda]+\cdots+[n], \end{aligned}$$

where $[\lambda]$ and the identity representation $[n]$ each appear once and only once, and $[\lambda_i]$ is the representation of the skew diagram component t_i.

When the operator $\prod R_{ik}$ is applied to the tableau s, the result is disregarded if

(1) Any row contains more symbols that a previous row, or

(2) Two symbols from the same row appear in the same column.

Robinson [5] proposes a systematization of the application of Young's raising operator R_{ik}, as follows. Suppose that the λ_1 symbols in the first row of t are a_i's, the λ_2 symbols in the second row are a_2's, and so on. We add the a_2's in succession from the left to t_1

with some in the first row, and the remainder in the second row, according to the operator:

$$(1-R_{12})^{-1} = 1 + R_{12} + R_{12}^2 + \cdots + R_{12}^{\lambda_2} + \cdots.$$

No operator R_{12}^i with $i > \lambda_2$ is meaningful when applied to t. Then we add the a_3's in all possible ways to each of the resulting diagrams subject to rules **1** and **2** above.

Each diagram so obtained is associated with an allowable operator from the product

$$(1-R_{12})^{-1}(1-R_{13})^{-1}(1-R_{23})^{-1}.$$

Continuing in this manner gives the reduction of the permutation representation as

$$[\lambda_1] \otimes [\lambda_2] \otimes \cdots \otimes [\lambda_h] = \prod (1-R_{ik})^{-1}[\lambda].$$

This equation may be inverted to give

$$[\lambda] = \prod (1-R_{ik})[\lambda_1] \otimes [\lambda_2] \otimes \cdots \otimes [\lambda_h].$$

Each operator raises a symbol from a specified row, and rows are disjoint. Therefore the restrictions **1** and **2** above do not apply to this equation. This is an induction on the identity representation of the subgroup S_λ to obtain a representation of S_n.

6.2 Lattice Permutations

In a previous section we described the reduction of the permutation representation $[\lambda_1] \otimes [\lambda_2] \otimes \ldots \otimes [\lambda_k]$ of S_n in terms of the operator R_{ij}. A more explicit alternative is to associate each of the f_λ standard skew tableaux t^λ of $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_k$ with a standard right tableau of an irreducible component. To do this, we firstly superimpose the rows of $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_k$ to yield f_λ tableaux t^λ of λ, of which only f^λ are standard. We form the association by denoting each symbol in the ith row of t^λ by a_i. Thus any arrangement of $1, 2, \cdots, n$ in t^λ defines a unique permutation π of the a_i. This permutation is defined by the $\lambda_1!\lambda_2!\ldots\lambda_k!$ tableaux obtained by rearranging the symbols in the rows of t^λ. If t^λ is a standard tableau, then the first r terms of π contains at least as many a_i's as a_{i+1}'s for all i and all r. Such a permutation is a lattice permutation. Conversely, each lattice permutation defines a unique standard tableau t^λ.

Example 6.2 *The lattice permutation corresponding to the standard tableau*

$$\begin{array}{lll} 1\ 3\ 5 & \text{--------} & a_1, \\ 2\ 4 & \text{--------} & a_2. \end{array}$$

is $a_1a_2a_1a_2a_1$. Thus the equivalence of the lattice property with standardness is obvious.

6.3 Skew Diagrams

Consider a diagram α and a diagram β which is superimposed upon α, with upper left hand corner upon upper left hand corner, such that β is contained entirely within α. The part of α not covered by β is called a skew diagram, denoted by α/β. The part of the rim of α beginning with the last node of any row and ending with the last node of an earlier column is called a skew hook.

The description of α/β in terms of α and β is not unique. It may consist of one or more disjoint constituents which may themselves be skew diagrams. It is possible to choose different α and β to represent the same skew diagram. However, in the context of this research, we will normally choose α and β such that β has normal shape.

We now introduce the operator B^{-1} which will annihilate those nodes of α belonging to β, setting

$$\alpha/\beta = B^{-1}\alpha = B^{-1} \mid [\alpha_i - i + j] \mid^{\bullet}$$

where

$$B^{-1} = (R_{01})^{\beta_1}(R_{02})^{\beta_2}(R_{03})^{\beta_3}\cdots,$$

and the dot exponent indicates the type of multiplication used. In this case it is the multiplication of the elements of the determinant, where addition and subtraction denote union and difference of tableau shapes respectively.

The operator $(R_{01})^{\beta_1}$ corresponds to subtracting β_1 nodes from each term in the first column of $\mid (\alpha_i - i + j) \mid^{\bullet}$. The operator $(R_{02})^{\beta_2}$ corresponds to subtracting β_2 nodes from each term in the second column, and so on.

The result of making these changes can again be expressed as a determinant in the corresponding representation, viz

$$(\alpha)/(\beta) = | \, (\alpha_i - i - \beta_j + j) \, |^{\bullet},$$

with degree

$$f^{\alpha/\beta} = n! \left| \frac{1}{(\alpha_i - i - \beta_j + j)!} \right| .$$

From the previous equations, we derive the important result that

$$(\alpha) = B \, | \, (\alpha_i - i - \beta_j + j) \, |^{\bullet},$$

where we assume that the operator affects only the factors arising from the first column.

Example 6.3 *If we consider the tableaux* $\alpha = (3,^2,1)$ *and* $\beta = (2,1)$*, then* α/β *is the skew tableau*

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{array} \Big/ \begin{array}{cc} \bullet & \bullet \\ \bullet & \\ & \end{array} = \begin{array}{ccc} & & \bullet \\ & \bullet & \bullet \\ \bullet & & \end{array}$$

Example 6.4 *This skew diagram may equally well be defined by* α/β*, where* $\alpha = (4,3,1)$

and $\beta = (3,1)$, *viz*

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \end{array} \Big/ \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & & \\ \bullet & & \end{array} = \begin{array}{cccc} & & & \bullet \\ & \bullet & \bullet & \\ \bullet & & & \end{array}$$

Example 6.5 *Alternatively, this skew diagram may be defined as* $\alpha/\beta = (4,3,3,1)/(3,3,1)$, *that is,*

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array} \Big/ \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \\ & & \end{array} = \begin{array}{cccc} & & & \bullet \\ & & & \\ & \bullet & \bullet & \\ \bullet & & & \end{array} = \begin{array}{cccc} & & & \bullet \\ & \bullet & \bullet & \\ \bullet & & & \end{array}$$

Example 6.6 *Using our previous Example 6.3, with* $\alpha = (3^2,1)$ *and* $\beta = (2,1)$, *then*

$$\begin{aligned} (\alpha)/(\beta) &= (3^2,1)/(2,1) \\ &= B^{-1} \mid (\alpha_i - i + j) \mid^{\bullet}, \end{aligned}$$

where

$$\begin{aligned} B^{-1} &= (R_{01})^{\beta_1}(R_{02})^{\beta_2} \\ &= (R_{01})^2(R_{02})^1 \\ &= R_{01}R_{01}R_{02}. \end{aligned}$$

Therefore

$$
\begin{aligned}
(\alpha)/(\beta) &= (R_{01})^2(R_{02}) \mid (\alpha_i - i + j) \mid^{\bullet} \\
&= \mid (\alpha_i - i - \beta_j + j) \mid^{\bullet} \\
&= \begin{vmatrix} (3-1-2+1) & (3-1-1+2) & (3-1-0+3) \\ (3-2-2+1) & (3-2-1+2) & (3-2-0+3) \\ (1-3-2+1) & (1-3-1+2) & (1-3-0+3) \end{vmatrix}^{\bullet} \\
&= \begin{vmatrix} (1) & (3) & (5) \\ (0) & (2) & (4) \\ (-3) & (-1) & (1) \end{vmatrix}^{\bullet},
\end{aligned}
$$

and the degree of (α/β) *is given by*

$$f^{\alpha/\beta} = 7! \left| \frac{1}{(\alpha_i - i - \beta_j + j)} \right|^{\bullet}.$$

In forming the determinant, addition and subtraction operators denote union and difference of tableau shapes respectively. We substitute the difference operator / where the meaning is obvious.

Example 6.7 *Consider the partition* $\lambda = (3,2,1)$ *of* $n = 6$. *We write*

$$\begin{aligned}
\lambda &= |\,(\lambda_i - i + j)\,|^{\bullet} \\
&= \begin{vmatrix} (3-1+1) & (3-1+2) & (3-1+3) \\ (2-2+1) & (2-2+2) & (2-2+3) \\ (1-3+1) & (1-3+2) & (1-3+3) \end{vmatrix}^{\bullet} \\
&= \begin{vmatrix} (3) & (4) & (5) \\ (1) & (2) & (3) \\ (-1) & (0) & (-1) \end{vmatrix}^{\bullet} \\
&= \begin{vmatrix} (r) & (r+1) & (r+2) \\ (s-1) & (s) & (s+1) \\ (t-2) & (t-1) & (t) \end{vmatrix}^{\bullet}, \qquad \textit{where } r = 3, s = 2, \textit{ and } t = 1.
\end{aligned}$$

$$
\begin{aligned}
(\alpha)/(\beta) &= \begin{vmatrix} (1) & (3) & (5) \\ (0) & (2) & (4) \\ (-3) & (-1) & (1) \end{vmatrix} \\
&= (1)\times((2)\times(1)+(1)\times(4))-(3)\times((0)\times(-1)-(-3)\times(2)) \\
&\quad +(5)\times((0)\times(-1)-(-3)\times(2)) \\
&= (2)\times(1)\times(1)+(4)\times(1)\times(1)-(3)\times(0)\times(-1)+(3)\times(2)\times(-3)+ \\
&\quad (5)\times(0)\times(-1)-(5)\times(2)\times(-3) \\
&= (2)\times(1)\times(1)+(1)\times(1)\times(0)+(3)\times(2)\times(-3)+(0)\times(-2)\times(-4) \\
&= (2)\times(1)\times(1)+(1)\times(1)\times(0)+(3)\times(0)\times(-7).
\end{aligned}
$$

It is convention to write 1 *for* (0) *and* 0 *for* $(-r)$*. This gives*

$$
\begin{aligned}
(\alpha)/(\beta) &= \begin{vmatrix} (1) & (3) & (5) \\ 1 & (2) & (4) \\ 0 & 0 & (1) \end{vmatrix}^{\bullet} \qquad (6.1) \\
&= (1)\times((2)\times(1)-0)-(3)\times((1)-0)+(5)\times(0-0) \\
&= (2)\times(1)\times(1)-(3)\times(1).
\end{aligned}
$$

Example 6.8 *The skew diagram* $(3^2,1)/(2,1)$ *is equally well defined as* $(4^2,2,1)/(3,2^2)$

and we write out its eight standard tableaux and their associated lattice permutations

$$\begin{array}{ccccccc} \;2 & \;1 & \;1 & & & & \\ 13 & 23 & 24 & \longrightarrow & a_1a_1a_2a_3, a_1a_2a_1a_3, a_1a_2a_3a_1 & : & (2,1^2) \\ 4 & 4 & 3 & & & & \end{array}$$

$$\begin{array}{cccccc} \;2 & \;3 & & & & \\ 14 & 14 & \longrightarrow & a_1a_1a_2a_2, a_1a_2a_1a_2 & : & (2^2) \\ 3 & 2 & & & & \end{array}$$

$$\begin{array}{ccccccc} \;3 & \;2 & \;1 & & & & \\ 24 & 34 & 34 & \longrightarrow & a_1a_1a_2a_3, a_1a_2a_1a_3, a_1a_2a_3a_1 & : & (2,1^2) \\ 1 & 1 & 2 & & & & \end{array}$$

so that

$$(3^2,1)/(2-1) = (2,1^2) + (2^2) + (3,1).$$

We also have

$$\begin{aligned} (3^2,1)/(2-1) &= \begin{vmatrix} (1) & (3) & (5) \\ 1 & (2) & (4) \\ 0 & 0 & (1) \end{vmatrix}^{\bullet} \\ &= (2)\times(1)\times(1) - (3)\times(1). \end{aligned}$$

Using the reduction $(3)\times(1)=(3,1)+(4)$. *We may write*

$$\begin{aligned}(3^2,1) &= \begin{vmatrix}(0) & (3) & (4)\\ (-1) & (2) & (3)\\ (-4) & (-1) & (0)\end{vmatrix}^{\bullet} + (3,1^2)\\ &= \begin{vmatrix}(3,1^2) & (3) & (4)\\ (2,1^2) & (2) & (3)\\ (0) & (0) & (1)\end{vmatrix}^{\bullet} = \begin{vmatrix}(3,1^2) & (3)\\ (2,1^2) & (2)\end{vmatrix}^{\bullet}.\end{aligned}$$

This gives

$$(3^2,1)/(2,1)=(2,1^2)+(2^2)+(3,1).$$

From the previous example

$$(2)\times(1)\times(1)=(2,1^2)+(2^2)+2(3,1)+(4).$$

Substituting gives

$$(3^2,1)/(2,1)=(2,1^2)+(2^2)+(2^2)+2(3,1)+(4)-(3)\times(1).$$

Applying the reduction $(3)\times(1)=(3,1)+(4)$ *gives*

$$\begin{aligned}(3^2,1)/(2,1) &= (2,1^2)+(2^2)+2(3,1)+(4)-(3,1)-(4)\\ &= (2,1^2)+(2^2)+(3,1).\end{aligned}$$

This shows that the two methods give the same result.

6.4 Young's Theorem for Skew Tableaux

To define a skew representation, we first introduce Young's theorem, which is identical to the Young-Yamanouchi basis of Chapter 5, except that tableaux are ordered according to dictionary order rather than last-letter order. We give the theorem again for completeness.

Theorem 6.1 ***Young's Theorem*** *([5], page 38.)(*

To construct the matrix representing $(r, r+1)$ in the irreducible representations $[\lambda]$, arrange the f^λ standard tableaux $\cdots, t^\lambda_\mu, \cdots, t^\lambda_\nu \cdots$ in dictionary order and set

(1) *1 in the leading diagonal where t^λ has r and $r+1$ in the same row;*

(2) *-1 in the leading diagonal where t^λ has r and $r+1$ in the same column;*

(3) *A quadratic matrix:*

$$\begin{bmatrix} -\rho & \sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & \rho \end{bmatrix}$$

at the intersection of the rows and columns corresponding to t^λ_μ and t^λ_ν where $\mu < \nu$ and t^λ_ν is obtained from t^λ_μ by interchanging r and $r+1$.

If r appears in the (i, j) position and $r+1$ appears in the (k, l) position of t^λ_u with $i < k, j > l$ then

$$\rho = (g_{ij} - g_{kl})^{-1} = [(j-i) - (l-k)]^{-1};$$

(4) *Zeros elsewhere.*

Young's theorem enables us to construct are representation corresponding to a direct product of tableaux associated with a direct product of subgroups.

Theorem 6.2 *The outer product* $[\mu].[\nu] = [\nu].[\mu]$*, where* $[\mu]$ *is any irreducible representation of* S_m *and* $[\nu]$ *is any irreducible representation of* S_n*, denotes that representation* S_{m+n} *induced by the irreducible representation* $[\mu] \otimes [\nu]$ *of* $S_m \times S_n$*. The matrices of the representation obtained by applying Young's Theorem to the standard tableaux of* $[\mu].[\nu]$*, setting* $\rho = 0$ *if* $r, r+1$ *belong to disjoint constituents.*

Now we consider again the restrictions of an irreducible representation $[\lambda]$ of S_n to a representation of a subgroup of the form

$$S_{n_1} \times S_{n_2} \times \cdots S_{n_s},$$

where $n_1 + n_2 + \cdots + n_s = n$.

To do this we use the skew representation $[\alpha]/[\beta]$, denoted here by $[\alpha/\beta]$.

Consider a sequence of Young diagrams

$$[\lambda] \supset [\lambda'] \supset [\lambda''] \cdots \supset [\lambda^{s-1}],$$

containing $n, n', n'', \cdots, n^{s-1}$ nodes respectively. This is represented symbolically as in Figure 6.1. If we set $n - n' = n_s, n' - n'' = n_{s-1}, \cdots, n^{s-1} = n_1$, then the skew diagrams

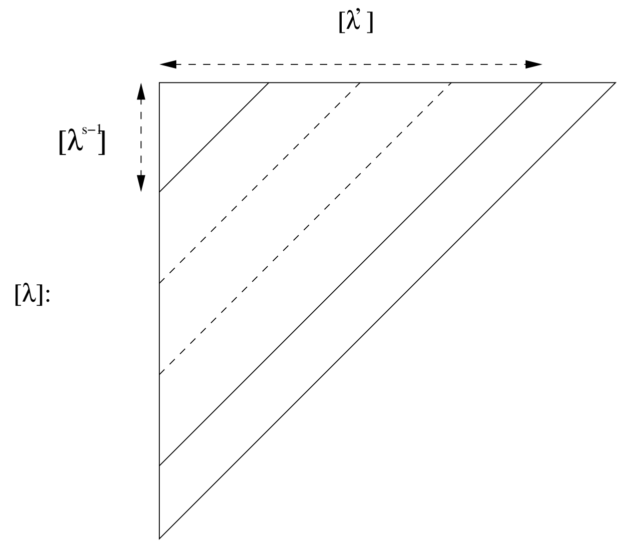


Figure 6.1: Symbolic representation of sequence of Young diagrams

$$(\lambda^{s-1}), (\lambda^{s-2}/\lambda^{s-1}), \cdots, (\lambda'/\lambda''), (\lambda/\lambda'),$$

give representations of $S_{n_1}, S_{n_2}, \cdots, S_{n_s}$, respectively. These representations are reducible in general, except for the first. Because the tableaux are all standard, the first n_1, symbols $1, 2, \cdots, n_1$ are associated with S_{n_1}, the next n_2 symbols are associated with S_{n_2}, and so on. Taking the transpositions according to Young's Theorem to give the matrices of the subgroups gives rise to a direct product

$$[\lambda^{s-1}] \otimes \cdots \otimes [\lambda - \lambda'].$$

Thus we are now able to define the skew representation. Considering all possible choices for the sequence of skew diagrams gives rise to a theorem as below.

Theorem 6.3 *If the irreducible representation* $[\lambda]$ *of* S_n *is restricted to the subgroup*

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_s},$$

then the resulting representation is a sum of tensor products

$$\sum [\lambda^{s-1}] \times \cdots \times [\lambda - \lambda'].$$

The summation is taken over all possible sequences of skew diagrams, and in general each term is reducible.

Example 6.9 *Consider the irreducible representation* $[3,2,1]$ *of* S_6 *and let us first restrict to the subgroup* $S_3 \times S_2 \times S_1$.

$$[3,2,1] \downarrow S_3 \times S_2 \times S_1 = [3] \otimes [2] \otimes [1] + [3] \otimes [1^2] \otimes [1] + 3[2,1] \otimes [2,1/1] \otimes [1] + [1^3] \otimes [2] \otimes [1] + [1^3] \otimes [1^2] \otimes [1].$$

Similarly, we have the following reductions:

$$[3,2,1] \downarrow S_3 \times S_2 = [3] \otimes [2,1] + [2,1] \otimes [3,2,1/2,1] + [1^3] \otimes [2,1],$$

$$[3,2,1] \downarrow S_4 \times S_1 \times S_1 = 2([3,1] \otimes [1] \otimes [1] + [2^1] \otimes [1] \otimes [1] + [2,1^2] \otimes [1] \otimes [1]),$$

$$[3,2,1] \downarrow S_3 \times S_1 = [3,2] \otimes [1] + [3,1^2] \otimes [1] + [2^2,1] \otimes [1].$$

Chapter 7

The Transition Matrix between Symmetric Group Bases

In this chapter, we introduce the contribution of our research, which is the determination of the matrix which transforms between symmetric group bases associated with representations of direct products of subgroups of S_n of the form

$$S_{(n-a),a} \times S_a \quad \text{and} \quad S_{(n-b),b} \times S_b.$$

To do this, we give outlines of two research papers published by Hamel et al. [7] and McAven et al.[8] in the Journal of Physics A, and one by Pan and Chen [13].

In the paper by Hamel et al. [7] the authors calculate the matrix which transforms the basis vectors of the Young-Yamanouchi basis into the basis vectors of its dual. To do this, the authors devise the representation matrices for both bases and then determine the transformation matrix. The dual basis is associated with the subgroup chain

$$S_1 \times S_{n-1} \supset S_1 \times S_1 \times S_{n-2} \supset \cdots \supset S_1 \times S_1 \times \cdots S_1.$$

The Young-Yamanouchi, or YY, basis is associated with the subgroup chain

$$S_{n-1} \times S_1 \supset S_{n-2} \times S_1 \times S_1 \supset \cdots \supset S_1 \times S_1 \times \cdots S_1.$$

The combinatorial technique of jeu de taquin is used to define the dual, or $\overline{YY}$, basis in terms of the YY symbols and the Young tableaux with which the basis vectors can be induced.

In the paper by McAven et al. [8] the authors consider matrices transforming between the standard Young-Yamanouchi basis of the symmetric group S_n and the basis where three nodes are removed together. The authors derive formulae for all such transformation coefficients. A choice of multiplicity separation is required when the three boxes removed are all non-adjacent. The multiplicity separation links the $S_n - S_{(n-3),3}-$ basis with the standard basis. The authors then discuss considerations which can be applied to obtain simple forms for the transformation coefficients and for the multiplicity separation. Some simple, natural separations are obtained. However, the authors show that the combinatorial and algebraic structure of the Littlewood-Richardson rule, also known as the pattern calculus, does not fix the separation.

The first paper by McAven et al.[7] provides an example of the split basis in the simplest case, that of removing one node from a tableau. As such, it provides an example of the transition matrix in the simplest case. We include a survey of this paper for this reason.

The second paper by McAven et al. [8] tackles the issue of multiplicity separation in the split-basis. We establish in Chapter 7 that the two crucial issues are multiplicity separation and the representation of the bridging transposition. It is the focus of this thesis to address the issue of multiplicity separation, leaving the problem of bridging transposition to future research. As the paper by McAven et al.[8] discusses the problem of multiplicity separation, it is of direct relevance to this thesis and has been discussed here.

Other papers are of relevance but have not been surveyed here. In the paper by McAven et al.[9] prove the block selective conjecture as a means to derive a representation for the bridging transposition. In a further paper McAven et al.[21] go on to use the block-selective conjecture to derive transformation coefficients, including for the bridging transposition. As the issue of the bridging transposition would be the topic of further research, these papers are only relevant to further research and are thus not surveyed here.

Chilla[22][23] also published two papers addressing the issue of subduction coefficients in the split basis. These two papers should also be considered in any future research.

7.1 Transformation Between the Young-Yamanouchi Basis And Its Dual

Hamel et al. [7] give the transformation matrix between the Young-Yamanouchi basis and its dual in their paper. The research is aimed at investigating representations of the symmetric group, S_n, and the associated matrices, characters and bases, which arise in the study of the many-electron problem in quantum mechanics.

The Young-Yamanouchi, or YY basis, occurs frequently as a basis associated with the subgroup chain

$$S_{n-1} \times S_1 \supset S_{n-2} \times S_1 \times S_1 \supset \cdots S_1 \times S_1 \times \cdots S_1$$

The subgroup chain associated with the Young-Yamanouchi basis is a chain of maximal subgroups. Thus the YY basis vectors for the irreducible representations (irreps) of S_n are also basis vectors for the chain of subgroups. The irreps of a direct product of subgroups can be written as the direct product of irreps of the factor groups. The only irrep of S_1 is the one dimensional unit matrix. Thus in the YY basis, the irreps of the subgroups can be labelled by the first factor of the subgroup. In this way, each basis vector can be associated with the irreps to which it belongs in $S_n,\ S_{n-1},\ S_{n-2},\cdots,$. This identification corresponds to the unique Young tableaux for which removal of the box labelled $n,\ n-1,\ n-2,\cdots$ yields Young tableaux corresponding to the subgroup. Thus the basis vectors for the YY basis are indexed by the complete set of Young tableaux.

A more general set of basis vectors corresponds to the basis

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_\ell},$$

where

$$n_1 + n_2 + \cdots + n_\ell = n.$$

In this basis, the matrices of subgroups are direct sums of tensor products of matrix irreps of the factor groups. The YY basis is a specific case of this more general basis, where $n_i = 1$ for all $i > 1$. The problem posed by the Hamel et al. [7] is to find the general form of the transformation matrix between two such bases, i.e. for the transformation between subgroup bases of the form

$$S_{m_1} \times S_{m_2} \times \cdots S_{m_k},$$

and

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_\ell},$$

where

$$m_1 + m_2 + \cdots + m_k = n_1 + n_2 + \cdots + n_\ell = n.$$

This problem has been the subject of considerable research[7],[8],[9],[13]. Since the symmetric group is generated by adjacent transpositions, it suffices to study the transformation between representations of these transpositions.

The jeu de taquin is a combinatorial technique which is used to construct the $\overline{YY}$ basis. The $\overline{YY}$ symbols are constructed using the following algorithm :-

Algorithm YS

YS.1 Set $x = 1$.

YS.2 Remove the box labelled x from the Young tableau.

YS.3 Fill the hole left by the removal using jeu de taquin.

YS.4 Write down the index of the row from which a box is removed after application of jeu de taquin.

YS.5 Set $x = x + 1$.

YS.6 If $x > n$, the algorithm terminates. Otherwise, go to Step YS.2.

The list of integers generated by this algorithm is the $\overline{YY}$ symbol.

Example 7.1 *Consider the tableau t of Example 5.8. The sequence of Young tableau and $\overline{YY}$ symbol values is*

$$
\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 4 & 5 \\ \hline 6 & \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|} \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1} 6 & \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|} \hline 4 & 5 \\ \hline 6 & \\ \cline{1-1} \end{array}
$$

$$(1) \qquad (12) \qquad (123)$$

$$
\begin{array}{|c|} \hline 5 \\ \hline 6 \\ \hline \end{array}
\qquad
\begin{array}{|c|} \hline 6 \\ \hline \end{array}
$$

$$(1231) \quad (12312) \quad (123121)$$

Definition 7.1: The companion tableau, $\hat{t}$ of t is defined to be the tableau such that the $\overline{YY}$ symbol of $\hat{t}$ is equal to the YY symbol of t. The companion relation is a symmetric relation, that is, the YY symbol of $\hat{t}$ is equal to the $\overline{YY}$ symbol of t.

Example 7.2 *Consider the tableau of Example 5.8. Note that the tableau*

1	3	6
2	5	
4		

has YY symbol 123121 *and $\overline{YY}$ symbol* 322111. *Therefore these two tableaux are companion tableaux.*

The list of tableaux, YY symbols and $\overline{YY}$ symbols is given below, with the tableaux arranged in dictionary order.

Tableau	YY Symbol	$\overline{YY}$ Symbol	Tableau	YY Symbol	$\overline{YY}$ Symbol
1 2 3 4 5 6	322111	123121	1 2 5 3 6 4	213211	132121
1 2 4 3 5 6	321211	121321	1 3 5 2 6 4	213121	312121
1 3 4 2 5 6	321121	211321	1 4 5 2 6 3	211321	321121
1 2 5 3 4 6	312211	231121	1 2 6 3 4 5	132211	231211
1 3 5 2 4 6	312121	213121	1 3 6 2 4 5	132121	213211
1 2 3 4 6 5	232111	123211	1 2 6 3 5 4	123211	232111
1 2 4 3 6 5	231211	132211	1 3 6 2 5 4	123121	322111
1 3 4 2 6 5	231121	312211	1 4 6 2 5 3	121321	321211

The $\overline{YY}$ representation matrix, $\overline{M}^{\lambda}_{(k-1,k)}$ for the transposition $(k-1,k)$ is defined as follows:- First arrange the tableax in last letter order. Then the matrix entries are calculated according to the following prescription.

(1) The matrix, $\overline{M}^{\lambda}_{(k-1,k)}$ has $+1$ in position (r,r) if, in the r^{th} $\overline{YY}$ symbol, the $(k-1)^{\text{th}}$ and k^{th} elements are identical.

(2) The matrix, $\overline{M}^{\lambda}_{(k-1,k)}$ has -1 in position (r,r) if in the r^{th} $\overline{YY}$ symbol, the $(k-1)^{\text{th}}$ element, α, is one more than the k^{th} element, β, and there does not exist another

$\overline{YY}$ symbol which is identical to this $\overline{YY}$ symbol except that its $(k-1)^{\text{th}}$ element is β and its k^{th} element is α.

(3) The matrix , $\overline{M}^{\lambda}_{(k-1,k)}$ has $-\rho$ in position (r,r), $\sqrt{1-\rho^2}$ in positions (r,s) and (s,r), and ρ in position (s,s), if $r<s$ and the r^{th} and s^{th} $\overline{YY}$ symbols are identical except that the $(k-1)^{\text{th}}$ element of the r^{th} $\overline{YY}$ symbol is the k^{th} elements of the s^{th} $\overline{YY}$ symbol and vice-versa.

(4) The matrix $\overline{M}^{\lambda}_{(k-1,k)}$, has zero in all other positions.

Example 7.3 *The* $\overline{YY}$ *representation matrices for the transpositions* (12), (23), (34), (45) *and* (56) *are shown below.*

$$[(1,2)] = \left|\begin{array}{ccccccccccccccc}
\frac{1}{2} & & & & \sqrt{\frac{3}{2}} & & & & & & & & & & \\
 & \frac{1}{2} & \sqrt{\frac{3}{2}} & & & & & & & & & & & & \\
 & \sqrt{\frac{3}{2}} & \frac{-1}{2} & & & & & & & & & & & & \\
 & & & \frac{1}{2} & & & & & & & \sqrt{\frac{3}{2}} & & & & \\
\sqrt{\frac{3}{2}} & & & & \frac{-1}{2} & & & & & & & & & & \\
 & & & & & \frac{1}{2} & & & & & & & & & \\
 & & & & & & \frac{1}{4} & \sqrt{\frac{15}{4}} & & & & & & & \\
 & & & & & & \sqrt{\frac{15}{4}} & \frac{-1}{4} & & & & & & & \\
 & & & & & & & & \frac{1}{4} & \sqrt{\frac{15}{4}} & & & & & \\
 & & & & & & & & \sqrt{\frac{15}{4}} & \frac{-1}{4} & & & & & \\
 & & & \sqrt{\frac{3}{2}} & & & & & & & \frac{-1}{2} & & & & \\
 & & & & & & & & & & \frac{1}{2} & \frac{-1}{2} & & & \sqrt{\frac{3}{2}} \\
 & & & & & \sqrt{\frac{3}{2}} & & & & & & & \frac{1}{2} & \sqrt{\frac{3}{2}} & \\
 & & & & & & & & & & & & \sqrt{\frac{3}{2}} & \frac{-1}{2} & \\
 & & & & & & & & & & & \sqrt{\frac{3}{2}} & & & \frac{-1}{2}
\end{array}\right|$$

$$
[(2,3)] = \left|\begin{array}{cccccccccccccccc}
\frac{1}{2} \\
& -1 \\
&& 1 \\
&&& \frac{-1}{4} & \sqrt{\frac{15}{4}} \\
&&& \sqrt{\frac{15}{4}} & \frac{1}{4} \\
&&&&& \frac{1}{2} & \sqrt{\frac{3}{2}} \\
&&&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&&&& \frac{-1}{2} \\
&&&&&&&& \frac{-1}{2} \\
&&&&&&&&& \frac{1}{2} & \sqrt{\frac{3}{2}} \\
&&&&&&&&& \sqrt{\frac{3}{2}} & \frac{-1}{2} \\
&&&&&&&&&&& \frac{-1}{4} & \sqrt{\frac{15}{4}} \\
&&&&&&&&&&& \sqrt{\frac{15}{4}} & \frac{1}{4} \\
&&&&&&&&&&&&& -1 \\
&&&&&&&&&&&&&& 1 \\
&&&&&&&&&&&&&&& \frac{-1}{2}
\end{array}\right|
$$

$$
[(3,4)] = \left|\begin{array}{cccccccccccccccc}
\frac{-1}{3} & \sqrt{\frac{8}{3}} \\
\sqrt{\frac{8}{3}} & \frac{1}{3} \\
&& \frac{1}{3} && \sqrt{\frac{8}{3}} \\
&&& 1 \\
&& \sqrt{\frac{8}{3}} && \frac{1}{3} \\
&&&&& -1 \\
&&&&&& 1 \\
&&&&&&& 1 \\
&&&&&&&& -1 \\
&&&&&&&&& -1 \\
&&&&&&&&&& 1 \\
&&&&&&&&&&& \frac{1}{3} && \sqrt{\frac{8}{3}} \\
&&&&&&&&&&&& -1 \\
&&&&&&&&&&& \sqrt{\frac{8}{3}} && \frac{-1}{3} \\
&&&&&&&&&&&&&& \frac{-1}{3} & \sqrt{\frac{8}{3}} \\
&&&&&&&&&&&&&& \sqrt{\frac{8}{3}} & \frac{1}{3}
\end{array}\right|
$$

$$[(4,5)] = \begin{vmatrix}
\frac{1}{2} & & & & & \sqrt{\frac{3}{2}} & & & & & & & & & & \\
& -1 & & & & & & & & & & & & & & \\
& & -1 & & & & & & & & & & & & & \\
& & & \frac{1}{2} & & & & & & & & & & & & \\
& & & & \frac{1}{2} & & & & & & & & & & & \\
\sqrt{\frac{3}{2}} & & & & & \frac{-1}{2} & & & & & & & & & & \\
& & & & & & \frac{-1}{2} & & \sqrt{\frac{3}{2}} & & & & & & & \\
& & & & & & & \frac{-1}{2} & & \sqrt{\frac{3}{2}} & & & & & & \\
& & & & & & \sqrt{\frac{3}{2}} & & \frac{-1}{2} & & & & & & & \\
& & & & & & & \sqrt{\frac{3}{2}} & & \frac{1}{2} & & & & & & \\
& & & & & & & & & & \frac{1}{2} & & & & & \sqrt{\frac{3}{2}} \\
& & & \sqrt{\frac{3}{2}} & & & & & & & & \frac{-1}{2} & & & & \\
& & & & \sqrt{\frac{3}{2}} & & & & & & & & \frac{-1}{2} & & & \\
& & & & & & & & & & & & & 1 & & \\
& & & & & & & & & & & & & & 1 & \\
& & & & & & & & & & \sqrt{\frac{3}{2}} & & & & & \frac{1}{2}
\end{vmatrix}$$

$$[(5,6)] = \begin{vmatrix}
-1 & & & & & & & & & & & & & & & \\
& -1 & & & & & & & & & & & & & & \\
& & -1 & & & & & & & & & & & & & \\
& & & -1 & & & & & & & & & & & & \\
& & & & -1 & & & & & & & & & & & \\
& & & & & 1 & & & & & & & & & & \\
& & & & & & 1 & & & & & & & & & \\
& & & & & & & 1 & & & & & & & & \\
& & & & & & & & -1 & & & & & & & \\
& & & & & & & & & -1 & & & & & & \\
& & & & & & & & & & -1 & & & & & \\
& & & & & & & & & & & 1 & & & & \\
& & & & & & & & & & & & 1 & & & \\
& & & & & & & & & & & & & 1 & & \\
& & & & & & & & & & & & & & 1 & \\
& & & & & & & & & & & & & & & 1
\end{vmatrix}$$

The representation matrices for the YY basis and the $\overline{YY}$ basis are related by a permutation matrix P, where

$$P\overline{M}^{\lambda}_{(k-1,k)}P^{-1} = M^{\lambda}_{(n-k+2,n-k+1)}.$$

We label the lists of YY and $\overline{YY}$ symbols by x and y, respectively. The permutation matrix P, which permutes the basis elements, is defined as

$$\begin{aligned} P_{i,j} &= \delta(x_i, y_j) \\ &= \delta(x_j, y_i), \end{aligned}$$

where δ is the Kronecker delta function, with

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

Thus the permutation matrix, P, maps each Young tableau to its companion tableau. We are now in a position to find the transformation matrix, T.

The transformation matrix, T, maps the representation matrix for $(k-1,k)$ in the YY basis, $M^{\lambda}_{(k-1,k)}$, to the representation matrix for $(k-1,k)$ in the $\overline{YY}$ basis, $\overline{M}^{\lambda}_{(k-1,k)}$. Thus

$$\overline{M}^{\lambda}_{(k-1,k)} = T^{-1}M^{\lambda}_{(k-1,k)}T \quad \text{for all } 1 < k \leq n.$$

The transformation matrix, T, may be expressed as the product, QP, of two matrices, P and Q. The matrix P is simply the permutation matrix defined above. It brings the matrix

$\overline{M}^{\lambda}_{(k-1,k)}$ to the same form as $M^{\lambda}_{(n-k+2,n-k+1)}$, as shown above. Therefore we have that

$$P^{-1}Q^{-1}M^{\lambda}_{(k-1,k)}QP = \overline{M}^{\lambda}_{(k-1,k)},$$

giving

$$\begin{aligned} Q^{-1}M^{\lambda}_{(k-1,k)}Q &= P\overline{M}^{\lambda}_{(k-1,k)}P^{-1} \\ &= M^{\lambda}_{(n-k+2,n-k+1)}. \end{aligned}$$

This equation effectively removes the representation matrix in the $\overline{YY}$ basis, $\overline{M}^{\lambda}_{(k-1,k)}$, from the computation of the transformation matrix. This enables the transformation matrix, Q, to be written in terms of mappings between representation matrices in the YY basis. Thus

$$Q^{-1}\overline{M}^{\lambda}_{(k-1,k)}Q = M^{\lambda}_{(n-k+2,n-k+1)}.$$

The Q matrix is the representation matrix that sends n, $n-1,\cdots 2,1$ to $1,2,\cdots,n-1,n$ respectively. Any permutation σ, can be written as a minimal length product of adjacent transpositions. This enables the Q matrix to be calculated directly from the representation matrices in the YY basis for the adjacent transpositions. The minimal length of the product is the number of inversions, that is, the number of distinct pairs (i,j) with $i<j$ such that $\sigma(i)>\sigma(j)$. We first define

$$d_i = card\{j \mid j>k \text{ where } \sigma(k)=i \text{ and } \sigma(j)<i\}.$$

$$
P = \begin{vmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{vmatrix}
$$

Figure 7.1: The P matrix for $\lambda = 3,2,1$.

Then the permutation σ, can be written as

$$\sigma = \cdots(\tau_{i-1}\tau_{i-2}\cdots\tau_{i-d_i})\cdots(\tau_{n-2}\tau_{n-3}\cdots\tau_{n-1-d_{n-1}})$$

$$(\tau_{n-1}\tau_{n-2}\cdots\tau_{n-d_n}),$$

where $\tau_i = (i, i+1)$ and the i^{th} contribution is included only if $d_i \geq 1$. In the case of Q, the length of the product will be $\binom{n}{2}$ and $d_i = i-1, 1 \leq i \leq n$. Specifically, then,

$$Q = \prod_{i=2}^{n} \prod_{j=i-1}^{1} (j, j+1).$$

Example 7.4 *With $\lambda = 3,2,1$, we have that*

$$Q = (16)(25)(34) = (12)(23)(12)(34)(23)(12)(45)(34)(23)(12)(56)(45)(34)(23)(12).$$

The matrix P and Q are shown in Figures 7.1 and 7.2, respectively.

$$
Q=\begin{vmatrix}
0 & \frac{1}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & 0 & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & \frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & 0 & \frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{3\sqrt{10}}{16} & 0 \\
\frac{1}{8\sqrt{2}} & \frac{1}{16} & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{-3}{16} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & \frac{3}{8} & \frac{\sqrt{5}}{16} & 0 & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{5}}{8} & \frac{\sqrt{-15}}{16} & \frac{\sqrt{-15}}{16} & 0 & \frac{3\sqrt{10}}{16} \\
\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8} & \frac{-3}{16} & \frac{5}{16} & 0 & -\frac{1}{8\sqrt{2}} & -\frac{1}{8} & \frac{\sqrt{3}}{16} & 0 & \frac{3\sqrt{5}}{16} & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} \\
\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8} & \frac{5}{16} & \frac{-3}{16} & 0 & \frac{3}{8\sqrt{2}} & \frac{3}{8} & \frac{5\sqrt{3}}{16} & 0 & \frac{\sqrt{5}}{16} & \frac{-\sqrt{10}}{16} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} \\
\frac{3}{8\sqrt{2}} & \frac{-3}{16} & 0 & 0 & \frac{5}{16} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & 0 & \frac{3\sqrt{5}}{16} & \frac{\sqrt{15}}{8} & -\frac{\sqrt{15}}{8\sqrt{2}} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{5}}{8} & \frac{\sqrt{10}}{16} \\
0 & \frac{\sqrt{3}}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & 0 & \frac{3}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{10}}{16} & 0 & \frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{10}}{16} & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & 0 \\
\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & \frac{-1}{8} & \frac{3}{8} & \frac{\sqrt{3}}{16} & \frac{3}{8\sqrt{2}} & \frac{3}{16} & -\frac{\sqrt{3}}{8} & -\frac{\sqrt{15}}{16} & 0 & -\frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8} & \frac{\sqrt{5}}{16} & -\frac{3\sqrt{5}}{16} & 0 & -\frac{\sqrt{15}}{8\sqrt{2}} \\
\frac{3}{8\sqrt{2}} & \frac{3}{8} & \frac{\sqrt{3}}{16} & \frac{5\sqrt{3}}{16} & 0 & -\frac{\sqrt{3}}{8\sqrt{2}} & -\frac{\sqrt{3}}{8} & -\frac{1}{16} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{3\sqrt{5}}{16} & 0 & 0 & \frac{\sqrt{5}}{16} & \frac{\sqrt{10}}{16} \\
\frac{\sqrt{10}}{16} & \frac{\sqrt{5}}{16} & 0 & 0 & -\frac{3\sqrt{5}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{16} & 0 & -\frac{1}{16} & -\frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8\sqrt{2}} & 0 & \frac{5\sqrt{3}}{16} & \frac{\sqrt{3}}{16} & \frac{3}{8} & \frac{3}{8\sqrt{2}} \\
-\frac{\sqrt{15}}{8\sqrt{2}} & 0 & -\frac{3\sqrt{5}}{16} & \frac{\sqrt{5}}{16} & \frac{\sqrt{15}}{8} & -\frac{\sqrt{10}}{16} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{3}}{8} & \frac{3}{16} & \frac{3}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & \frac{3}{8} & \frac{-1}{8} & \frac{\sqrt{3}}{16} & \frac{\sqrt{3}}{8\sqrt{2}} \\
0 & -\frac{\sqrt{15}}{8\sqrt{2}} & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & 0 & -\frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & 0 & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & 0 \\
\frac{\sqrt{10}}{16} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{16} & 0 & -\frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{15}}{8} & \frac{3\sqrt{5}}{16} & 0 & \frac{\sqrt{3}}{16} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{5}{16} & 0 & 0 & \frac{-3}{16} & \frac{3}{8\sqrt{2}} \\
-\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & \frac{-\sqrt{10}}{16} & \frac{\sqrt{5}}{16} & 0 & \frac{5\sqrt{3}}{16} & \frac{3}{8} & \frac{3}{8\sqrt{2}} & 0 & \frac{-3}{16} & \frac{5}{16} & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8\sqrt{2}} \\
-\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & \frac{3\sqrt{10}}{16} & \frac{3\sqrt{5}}{16} & 0 & \frac{\sqrt{3}}{16} & -\frac{1}{8} & -\frac{1}{8\sqrt{2}} & 0 & \frac{5}{16} & \frac{-3}{16} & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8\sqrt{2}} \\
\frac{3\sqrt{10}}{16} & 0 & \frac{\sqrt{-15}}{16} & \frac{\sqrt{-15}}{16} & \frac{\sqrt{5}}{8} & \frac{\sqrt{15}}{8\sqrt{2}} & 0 & \frac{\sqrt{5}}{16} & \frac{3}{8} & \frac{\sqrt{3}}{16} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{-3}{16} & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & \frac{1}{16} & \frac{1}{8\sqrt{2}} \\
0 & \frac{3\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{10}}{16} & 0 & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{10}}{16} & \frac{3}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & 0 & \frac{3}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & 0
\end{vmatrix}
$$

Figure 7.2: The Q matrix for $\lambda = 3, 2, 1$ calculated using Matlab

Hamel et al. [7] have presented a simple, straightforward method for calculating the transformation matrix between the YY basis and its dual, the $\overline{YY}$ basis. It is the goal of the research to generalize this method to finding the transformation matrix between more general bases of symmetric, for example, between two bases of S_n of the form

$$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_\ell} \quad \text{with} \quad n_1 + n_2 + \cdots n_\ell = n.$$

7.2 Multiplicity Separation In Symmetric Group Transformation Coefficients

McAven et al.[8] develop their initial research to consider multiplicity separation in symmetric group transformation coefficients. The matrices for the standard Young-Yamanouchi basis of the symmetric group S_n were described in Chapter 5. McAven et al.[8] now consider the matrices transforming between this basis and the basis where three nodes are removed from the Young tableau together. This is known as the $S_n - S_{n-3,3}$ basis. They develop closed formulae for all such transformation coefficients. When the three removed nodes are all non-adjacent, a choice of multiplicity separation is needed. The multiplicity separation links the $S_n - S_{n-3,3}$ basis vectors with the basis vectors of the standard basis.

The transition matrix transforms between one basis adapted to :-

$$\text{(i) } S_n, \text{ (ii) } S_a \times S_b, \text{ and (iii) } S_a \times S_e \times S_d,$$

and a second basis adapted to :-

$$\text{(i) } S_n, \text{ (ii) } S_c \times S_d, \text{ and (iii) } S_a \times S_e \times S_d,$$

where $a+b=a+e+d=c+d=n$.

The transition matrix can be expressed in terms of the transformation coefficients between the standard Young-Yamanouchi basis and a second basis adapted to the direct product subgroup, $S_a \times S_b$. This is known as a split-basis, and is written as the $S_n - S_{a,b}$ basis.

The first symmetric group in which a product multiplicity occurs is S_6. Specifically, the decomposition into the direct product subgroup, $S_3 \times S_3$, contains multiplicity two terms. The authors use the linear equation method to derive the general solution for $S_n - S_{n-b,b}$ bases for $b=3$. This method allows the derivation of the formulae for the coefficients associated with the shapes remaining after removing the S_a irreducible representation from the S_n irreducible representation. McAven et al.[8] give a general solution for the case where product multiplicities occur, before a choice of multiplicity separation is made.

7.2.1 A Linear Equation Method

Pan and Chen [13] have tabulated matrices which transform from the standard basis to the split-bases, for particular cases. Pan and Chen adapt the split-bases of the Hecke algebras. The Hecke algebra, $H_n(q)$, is a generalization of the symmetric group algebra. For the symmetric group, adjacent transpositions square to one. For the Hecke algebra, an additional function, given by

$$g_i^2 = g_i(q - q^{-1}) + 1,$$

with g_i an adjacent transposition, is used. The special case where $q = 1$ yields the symmetric group relation.

McAven et al. [8] formalize the method of Pan and Chen applied to the calculation of transformations between the $H_n(q)-$basis and a split $H_n(q) - H_{n_1,n_2}(q)-$basis. To do this, they generalize the basis notation introduced in the previous section to suit the Hecke algebra.

Let $\{g_1, g_2, \cdots, g_{n-1}\}$ be a set of adjacent transpositions which generate $H_n(q)$. Further, let $H_{n_1}(q)$ and $H_{n_2}(q)$ be sub-algebras generated by the subsets of transpositions

$$\{g_1, g_2, \cdots, g_{n_1-1}\} \quad \text{and} \quad \{g_{n_1+1}, g_{n_1+2}, \cdots, g_{n-1}\},$$

respectively. The expansion of the

$$H_n(q) - H_{n_1,n_2}(q) - \text{basis},$$

in terms of the $H_n(q)$ basis is given by Pan and Chen as

$$\langle [\lambda], \tau^{\alpha}_{m_1}{}^{\beta}_{m_2} \rangle_q = \sum_m \langle |^{[\lambda]}_m \rangle_q \langle^{[\lambda]}_m | [\lambda], \tau^{\alpha}_{m_1}{}^{\beta}_{m_2} \rangle_q. \tag{7.1}$$

Pan and Chen then derive two sets of linear equations using the two sets of generators for the Hecke sub-algebras. An equivalent means of deriving the sets of equations exists. To do this, consider the left and right actions of an operator between a state labelled by a standard basis vector and a state labelled by a split-basis vector.

$$\begin{aligned} \langle[\lambda],\tau^{\alpha\ \beta}_{m_1\,m_2}|g_{i,j}|^{[\lambda]}_{m}\rangle_q &= \langle[\lambda],\tau^{\alpha\ \beta}_{m_1\,m_2}|(g_{i,j}|^{[\lambda]}_{m}\rangle_q) \\ &= (\langle[\lambda],\tau^{\alpha\ \beta}_{m_1\,m_2}|g_{i,j})|^{[\lambda]}_{m}\rangle_q, \end{aligned} \tag{7.2}$$

The actions of the generators depend on various axial distances. This distance in the first tableau of the split-basis is the distance from i to $i+1$, (d_{1i}). In the second tableau of the split-basis it is the distance from j to $j+1$, (d_{2j}). In the standard basis it is the distance from i to $i+1$ and from j to $j+1$ in the standard tableau.

The first set of equations obtained by Pan and Chen have not been suitably reduced. This is because the axial distance from i to $i+1$ in the standard basis (d_i) is the same as the axial distance from i to $i+1$ in the split-basis, (d_{1i}). Thus in the first set equations of Pan and Chen, the left-hand side vanishes and the coefficients on the right-hand side are equal to within a sign. This gives

$$\langle^{[\lambda]}_{m'}|[\lambda],\tau^{\alpha\ \beta}_{m_1\,m_2}\rangle_q = \pm\langle^{[\lambda]}_{m}|[\lambda],\tau^{\alpha\ \beta}_{m'_1\,m_2}\rangle_q. \tag{7.3}$$

The transformation coefficients are independent of m_1.

The second set of equations of Pan and Chen is

$$\left(\frac{q^{d_j}}{[d_j]} - \frac{q^{d_j^b}}{[d_j^b]}\right) |{}^{[\lambda]}_{m}\rangle_q \langle{}^{[\lambda]}_{m}|[\lambda], \tau^{\alpha\ \beta}_{m_1\, m_2}\rangle_q =$$
$$\left(\frac{[d_j^h+1][d_j^b-1]}{[d_j^b]^2}\right)^{\frac{1}{2}} |{}^{[\lambda]}_{m}\rangle_q \langle{}^{[\lambda]}_{m}|[\lambda], \tau^{\alpha\ \beta}_{m_1\, m_2'}\rangle_q$$
$$- \left(\frac{[d_j+1][d_j-1]}{[d_j]^2}\right)^{\frac{1}{2}} |{}^{[\lambda]}_{m}\rangle_q \langle{}^{[\lambda]}_{m'}|[\lambda], \tau^{\alpha\ \beta}_{m_1\, m_2}\rangle_q. \tag{7.4}$$

Pan and Chen also formulate orthonormality conditions on the transformation coefficients,

$$\sum_m \langle{}^{[\lambda]}_{m}|[\lambda], \tau^{\alpha\ \beta}_{m_1\, m_2'}\rangle_q^2 = 1. \tag{7.5}$$

This system may be solved in blocks, where the blocks are labelled by αm_1. The individual m_1 are distinct with orthogonality satisfied, so the blocks may be solved separately. Within each block there are sub-blocks of rows which are related by the generators of the second subgroup. However, as the sub-blocks are orthonormal, the complete block may be calculated.

McAven et al.[8] then construct a computational method of formally constructing the general system of linear equations. The matrix X is the matrix for the general system of linear equations, that is, by setting $XL = 0$ we can obtain a homogeneous set of equations where L is defined below. Blocks can be solved independently, so the matrix is constructed for each block. McAven et al. construct the matrix X for the transformation coefficients appearing in the block corresponding to the second factor group in the direct product subgroup.

All of the tableau pairs labelling this block have the same first tableau, but they differ in the second tableau. Removal of the shape associated with the irreducible representation α from the upper left-hand corner of the shape associated with the irreducible representation λ leaves a skew shape. We denote the number of ways of filling this shape by N. The number N is found from Young tableau theorems and is equal to the dimension of the block.

This list of transformation coefficients is ordered first on the tableau pairs and then on the standard tableaux.

Example 7.5 *We are given a list of four standard tableaux*

$$U = \{A, B, C, D\},$$

and an associated list of four tableau pairs

$$V = \{A', B', C', D'\}.$$

Then our ordered list is

$$L = \{< A'|A >, < A'|B >, < A'|C >, < A'|D >,$$
$$< B'|A >, < B'|B >, < B'|C >, < B'|D >,$$
$$< C'|A >, < C'|B >, < C'|C >, < C'|D >,$$
$$< D'|A >, < D'|B >, < D'|C >, < D'|D > \}.$$

The basis tableaux in both bases are ordered in U and in V according to the conventions described in Section 5.2.

The first row of X corresponds to the two-way expansion of $< A'|g_a|A >$ as in Equation 7.2. The next $m^2 - 1$ rows relate to the expansion associated with the generator g_a of the other transformation coefficients in L where $m = dim(U) = dim(V)$. Then we move to the next generator, g_{a+1}, and return to the start of L. We know from Equation 7.4 that at most three entries in each row of X will be non-zero, so that X is sparse.

We consider the general transformation from the S_n-basis to the $S_n - S_{n-b,b}$ split basis. The transformation matrix is associated with the shape of the tableau obtained by removing $n - b$ nodes from the basis tableaux of the S_n-basis. This produces a tableau pair, having shapes α and λ/α respectively. The first is a standard tableau associated with the irreducible representations of the S_{n-b} subgroup. The second shape is skew and is associated with non-standard irreducible representation of S_b. The second tableau may be brought to normal shape using jeu de taquin. This is associated with an irreducible representation indexed by tableau pairs of the split-bases.

The transformation coefficient is the same for each basis vector of the first irreducible representation. Thus the transformation coefficient matrix splits into blocks of dimension $|\lambda/\alpha|$ of the skew shape remaining after removal of the nodes. The blocks are indexed by the tableaux in the standard basis, and by the tableau pairs in the second basis.

Let m be the dimension of the skew shape λ/α. The transformation coefficient matrix will contain m^2 rows for each generator, or adjacent transposition, of S_b. These correspond to indexing the list L, firstly on the tableau pair of the split basis and secondly on tableaux of the standard basis. Thus there exists a block for each adjacent transposition. Note that there are m^2 entries in the list L.

These correspond to the pairing of each standard tableau with each tableau pair. We must express the transformation coefficients for each pairing, for each generator. This accounts for the m^2 rows in the matrix for each generator.

The transformation coefficients correspond to an operator from a state labelled by the standard basis to a state labelled by the split-basis.

The dimension of the irreducible representation is the number of tableau and therefore, tableau pairs, given by m. This determines the size of the matrix representation of the generator in the irreducible representation, and therefore the size of each sub-block.

The transformation coefficient matrix performs the transformation

$$XM^{\lambda}(g_i)X^{-1} = M^{\lambda}_{a,b}(g_i),$$

for each generator g_i. We can index into the matrix X firstly on the tableau pairs and secondly on the tableaux to find the coefficient which transforms from the matrix $M^{\lambda}(g_i)$ for the generator, g_i, in the irreducible representation for the partition, λ, to the the matrix $M^{\lambda}_{a,b}(g_i)$ for the generator g_i, in the irreducible representation in the split basis.

When multiplicity separation occurs, non-square blocks of transformation coefficients may arise. In this case, the redundant terms in X may be removed by applying Gauss-Jordan elimination to obtain reduced row echelon form. The algebraic conditions are the same for transformation coefficients differing only in the product multiplicity. This implies that only a sub-block associated with one multiplicity needs to be calculated. The solution for the other multiplicities may be obtained by a relabelling of variables, with consideration of orthonormality conditions.

With X constructed, we solve the homogeneous system of equations

$$XL = 0.$$

together with the orthonormality conditions described by Equation 7.5. In the cases without product multiplicities, we need only make phase choices.

7.2.2 Formulae For Removing Three Nodes

McAven et al. [8] considered removing three nodes from a Young tableau, first when there is no multiplicity, and then for the case where multiplicity occurs. They do this by considering the general transformation from the standard S_n-basis to the split $S_n - S_{n-b,b}$ basis. The transformation matrix may be split into cases associated with the shape of the tableau of the S_n-basis. Removal of these nodes yields two shapes, α, and a skew shape λ/α. The first shape is standard and is associated with irreducible representations of the S_{n-b} subgroup. The second shape is associated with the non-standard irreducible representations of the S_b subgroup. The second shape may be standardized using jeu de taquin to give the second irreducible representation of the pair labelling. The shape of the first representation is invariant under the action of permutations. Schur's lemma implies that the transformation coefficient is the same for each basis vector of this representation. In this way, the transformation matrix splits into two blocks. The blocks are of dimension, $|\lambda/\alpha|$, of the skew shape of the second tableau. The basis vectors are obtained in the manner described in Section 7.2.1.

There are two irreps of dimension one. These are $[b]$ and $[1^b]$. They always give rise to a single 1×1 block with a simple phase freedom.

Now consider removal of two nodes. These two nodes are n and $(n-1)$. The three cases are shown below.

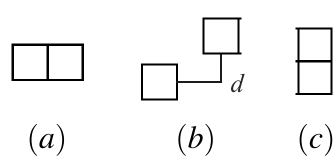


Figure 7.3: Removal of two nodes from a tableau.

Cases (a) and (c) have transformation coefficients ± 1.

To investigate case (b), we order the basis tableaux as described in Chapter 5. Label the case alphabetically according to the order. Equation 7.2 gives the two-way expansion of the entries on the diagonal of the 2×2 transformation matrix, given by

$$< A'|(n-1,1)|A >=< A'|A >= \frac{-1}{d} < A'|A > + \frac{\sqrt{d^2-1}}{d} < A'|B >.$$

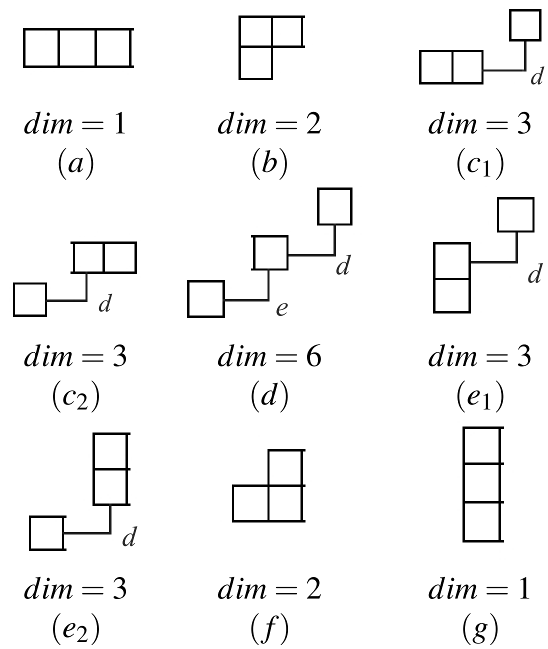


Figure 7.4: The skew shapes remaining after removal of $(n-3)$ nodes, showing axial distances d and e.

In this equation, d is the absolute value of the axial distance from the node $(n-1)$ to the node n, in the tableau labelled A. Using this value of d, we have

$$< B'|(n-1,1)|B >= - < B'|B >= \frac{-1}{d} < B'|A > + \frac{\sqrt{d^2-1}}{d} < A'|B >.$$

We use Θ to denote a phase ±1. Applying orthonormality of the basis tableaux gives

$$\begin{vmatrix} \Theta\sqrt{\frac{(d-1)}{2d}} & \Theta\sqrt{\frac{(d+1)}{2d}} \\ -\Theta\sqrt{\frac{(d+1)}{2d}} & \Theta\sqrt{\frac{(d-1)}{2d}} \end{vmatrix}.$$

Next, consider the removal of three nodes. The nine (9) cases are shown in Figure 7.4.

McAven et al.[8] used the package MAPLE to implement the linear equation method of Pan and Chen[13]. Thus they derive a general formula for this case. Two phases, Θ and ϕ, occur here. The cases are:

Case (a)

$$\text{Completely symmetric, } \Theta. \tag{7.6}$$

Case(b)

$$\begin{vmatrix} \Theta & 0 \\ 0 & \Theta \end{vmatrix} \tag{7.7}$$

Case(c_1)

$$\begin{vmatrix} \Theta\sqrt{\frac{d_-}{3d_+}} & \Theta\sqrt{\frac{d_-d_{++}}{3dd_+}} & \Theta\sqrt{\frac{d_{++}}{3d}} \\ -\phi\sqrt{\frac{d_{++}}{6d_+}} & -\phi\sqrt{\frac{d_{++}}{6dd_+}} & \phi\sqrt{\frac{2d_-}{3d}} \\ -\phi\sqrt{\frac{d_{++}}{2d_+}} & \phi\sqrt{\frac{d}{2d_+}} & 0 \end{vmatrix} \tag{7.8}$$

Case(c_2)

$$\begin{vmatrix} \Theta\sqrt{\frac{d_-}{3d_+}} & \Theta\sqrt{\frac{d_-d_{++}}{3dd_+}} & \Theta\sqrt{\frac{d_{++}}{3d}} \\ -\phi\sqrt{\frac{2d_{++}}{3d_+}} & -\phi\sqrt{\frac{d_-}{6d_+d}} & \phi\sqrt{\frac{d_-}{6d}} \\ 0 & -\phi\sqrt{\frac{d_+}{2d}} & \phi\sqrt{\frac{d_-}{2d}} \end{vmatrix} \tag{7.9}$$

Case(d)

$$\begin{vmatrix} \Theta\sqrt{\frac{d_-e_-f_-}{6def}} & \Theta\sqrt{\frac{d_-e_+f_-}{6def}} & \Theta\sqrt{\frac{d_+e_-f_-}{6def}} & \Theta\sqrt{\frac{d_-e_+f_+}{6def}} & \Theta\sqrt{\frac{d_+e_-f_+}{6def}} & \Theta\sqrt{\frac{d_+e_+f_+}{6def}} \\ \phi\sqrt{\frac{d_+e_+f_+}{6def}} & -\phi\sqrt{\frac{d_+e_-f_+}{6def}} & -\phi\sqrt{\frac{d_-e_+f_+}{6def}} & \phi\sqrt{\frac{d_+e_-f_-}{6def}} & \phi\sqrt{\frac{d_-e_+f_-}{6def}} & -\phi\sqrt{\frac{d_-e_-f_-}{6def}} \end{vmatrix} \tag{7.10}$$

Case(e_1)

$$\begin{vmatrix} \Theta\sqrt{\frac{d_-}{2d}} & \Theta\sqrt{\frac{d_+}{2d}} & 0 \\ -\Theta\sqrt{\frac{d_-}{6d}} & \Theta\frac{d_-}{\sqrt{6d_+d}} & \Theta\sqrt{\frac{2d_{++}}{3d_+}} \\ \phi\sqrt{\frac{d++}{3d}} & -\phi\sqrt{\frac{d_-d_{++}}{3d_+d}} & \phi\sqrt{\frac{d_-}{3d_+}} \end{vmatrix} \tag{7.11}$$

Case(e_2)

$$\begin{vmatrix} 0 & -\Theta\sqrt{\frac{d}{2d_+}} & -\Theta\sqrt{\frac{d_{++}}{2d_+}} \\ -\Theta\sqrt{\frac{2d_-}{3d}} & -\Theta\sqrt{\frac{d_{++}}{6d_+d}} & \phi\sqrt{\frac{d_{++}}{6d_+}} \\ \phi\sqrt{\frac{d++}{3d}} & -\phi\sqrt{\frac{d_-d_{++}}{3d_+}} & \phi\sqrt{\frac{d_-}{3d_+}} \end{vmatrix} \tag{7.12}$$

Case(f)

$$\begin{vmatrix} \frac{\Theta}{2} & \frac{\sqrt{3}\Theta}{2} \\ \frac{\sqrt{3}\Theta}{2} & \frac{-\Theta}{2} \end{vmatrix} \tag{7.13}$$

Case(g)

$$\text{Completely antisymmetric, } \Theta. \tag{7.14}$$

In case (d) we give just the top and bottom row from the 6x6 block.

In these coefficients, there are the two basic axial distances, as shown in Figure 7.4, d and e, and also augmented hook lengths, $f = d+e$, $d_+ = d-1$, $d_{++} = d+2$, and $d_{--} = d-2$. Two phases, Θ and ϕ, occur in the multiplicity free cases.

In the matrices above, the four central rows of case (d) have multiplicity separation. The system of equations includes three orthonormality equations. Therefore, three independent choices of phase exist. These are denoted by Θ, ϕ and ψ. One free factor governs multiplicity separation. We express all coefficients in terms of two related variables, x and y, with

$$y = rx.$$

The variable x and y are not independent. The authors recommend choosing x and y to determine a desirable multiplicity separation.

This gives

$$x = \frac{1}{\sqrt{6def(2de+d-e+1)(1+3d_+d_-e_+e_-f_+f_-r^2)}}. \tag{7.15}$$

This equation has many solutions. We use a_{ij} to denote the element in the i^{th} row and j^{th} column of the four central rows of (d). Thus the solutions are :-

$$a_{11} = \sqrt{d_-/d_+}a_{13}$$

$$a_{12} = \sqrt{f_-/f_+}a_{14}$$

$$a_{13} = -d_+\sqrt{f_-}[\Theta e_{++}x + 3\psi d_-e_+e_-f_+y]$$

$$a_{14} = \sqrt{d_+d_-e_+e_-f_+}[2\Theta x - 3\psi f_+f_-y]$$

$$a_{15} = \sqrt{e_-/e_+}a_{16}$$

$$a_{16} = \sqrt{e_+e_-f_+}[-\Theta d_{-}x + 3\psi d_+d_-e_+f_-y]$$

$$a_{21} = \sqrt{d_-/d_+}a_{23}$$

$$a_{22} = \sqrt{f_-/f_+}a_{24}$$

$$a_{23} = -\sqrt{3d_+d_-e_+e_-f_+}[x + \phi d_+e_{++}f_-y]$$

$$a_{24} = f_+\sqrt{3f_-}[-x + 2\phi d_+ d_- e_+ e_- y]$$

$$a_{25} = \sqrt{e_- e_+} a_{26}$$

$$a_{26} = e_+\sqrt{3d_+ d_- f_-}[x - \phi d_{-} e_- f_+ y]$$

$$a_{31} = -\sqrt{d_+/d_-} a_{33}$$

$$a_{32} = -\sqrt{f_+/f_-} a_{34}$$

$$a_{33} = d_-\sqrt{3f_-}[-\Theta ex + \psi d_+ e_+ e_- f_+ y]$$

$$a_{34} = \psi(2de + d - e + 1) f_-\sqrt{3d_+ d_- e_+ e_- f_+ y}$$

$$a_{35} = -\sqrt{e_+/e_-} a_{36}$$

$$a_{36} = \sqrt{3e_+ e_- f_+}[\Theta dx + \psi d_+ d_- e_- f_- y]$$

$$a_{41} = -\sqrt{d_+/d_-} a_{43}$$

$$a_{42} = -\sqrt{f_+/f_-} a_{44}$$

$$a_{43} = \sqrt{d_+ d_- e_+ e_- f_+}[x - 3\phi d_- e f_- y]$$

$$a_{44} = (2de + d - e + 1)\sqrt{f_-} x$$

$$a_{45} = -\sqrt{e_+/e_-} a_{46}$$

$$a_{46} = e_-\sqrt{d_+ d_- f_-}[x + 3\phi d e_+ f_+ y].$$

7.2.3 Choices of Phase and Multiplicity Separation

In order to find the simplest and most natural form for the symmetric group transformation coefficients, Hamel et al.[7] make the following considerations.

(I) The transformation coefficients can be chosen to be real.

(II) The general formulae obtained in the previous sections depend only on the axial distances, and are independent of n. Therefore the phases and multiplicity separation should be chosen independent of n.

(III) In the cases where either of the axial distances d or e is unity, the multiplicity is lifted. The expression for the multiplicity two coefficients must reduce to the multiplicity free solutions.

(IV) The multiplicity separations should be chosen so that a maximal number of zero coefficients is obtained.

(V) It is desirable to have the coefficients expressible as a single surd of the form $a\sqrt{b}/c$, with a, b and c integers.

(VI) We require that the only prime numbers which occur in the surd are as small as possible relative to d and e.

7.2.4 Comments

The aims of McAven et al. [8] were as follows.

(i) The bridging transposition in the split basis cannot be calculated as directly as the other transpositions.

(ii) General formula for the split-standard transformation coefficients are not available.

The authors sought to extend the existing two node formulae to the case of three nodes.

(iii) Pan and Chen [13] make different multiplicity choices for the first situation in which such a choice is necessary.

(iv) They also aimed to determine if the Littlewood-Richardson rule could yield a canonical multiplicity separation.

McAven et al. [8] presented an explicit formula for the removal of three nodes. There are nine separate cases. There are distinguished by the skew shape remaining after the removal of $(n-3)$ nodes from the left. The authors obtained the general multiplicity two solutions for the $S_n - S_{n-3,3}$-basis. They listed six considerations which must be made in choosing phases and multiplicity separations. The simpler separations corresponded to one of twelve zero conditions. These occurred in pair linked by relabelling the multiplicity. Two of these six pairs matched the solutions of degenerate cases. McAven et al. [8] chose a solution which simplified the form of phases and magnitude.

McAven et al. [8] claimed that the Littlewood-Richardson rule does not lead to a specific separation. When no multiplicities exist, the Littlewood-Richardson rule gives the pattern relations between the split and standard bases. These pattern relations do not give a canonical set of basis functions for the bases labelled by multiplicity labels. Therefore, criteria beyond the Littlewood-Richardson rule must be applied.

7.3 Conclusion

In this chapter, we have surveyed two papers by Hamel et al.[7] and McAven et al.[8] on the calculation of the transformation matrix between symmetric group bases.

In Hamel et al. [7] the authors calculated the matrix which transforms the basis functions of the Young-Yamanouchi basis into the basis function of its dual. They first derived the representation matrices for both bases, and then calculated the transformation matrix. This is a very specific case of the more general problem of finding the transformation matrices between any symmetrical group bases.

In McAven et al.[8] they found the transformation matrices between the standard Young-Yamanouchi basis and the basis where three boxes are removed together. Thus McAven et al.[8] consider a more general case of transforming between symmetric group bases.

In the next chapter of this thesis, we endeavor to continue the research into the general problem of calculating the transformation matrix between symmetric group bases.

Chapter 8

The Decomposition of Tableaux Into Skew Tableaux

In this chapter, the author of this thesis develops the research by Hamel et al. [7] and McAven et al. [8] which is given in Chapter 7. The author of this thesis examines the decomposition of a tableau into component tableaux. The author of this thesis also consider the justification of the component tableau into normal tableaux.

The author of this thesis's approach is to apply set-theoretic and combinatorial concepts to the decomposition of a tableau. In this way, I attempt to establish a formal and rigorous mathematical framework in which further research may be undertaken.

Except where otherwise noted, all definitions, theorems and proofs in this chapter are my original work. Definitions, theorems and proofs marked (†) are the authors original work. To the best of my knowledge, they do not currently exist in the literature on tableaux.

8.1 An Ordered Decomposition of a Tableaux

In Chapter 7, we described the research undertaken by McAven et al. into the transition matrix between symmetric group bases. Primary to this research is the splitting of Young tableaux into component tableaux. In its most general formulation, this involves the removal of boxes from Young tableaux in arbitrary order. In this section, we attempt to place such removal of boxes into a generalized mathematical framework, so that the concepts and principles of algebra may be applied to the calculation of the transition matrix.

A Young tableau is a special case of a partial skew tableau. Recall that the latter has skew shape and distinct entries whose rows and columns increase. Our first definition formalizes the splitting of a partial skew tableau into component tableau. This definition encompasses the removal of nodes from a tableau as considered by McAven et al.

Though McAven et al. eventually use jeu de taquin to normalize the skew tableaux obtained from ordered decompositions, we consider the latter separately for two reasons. First, we wish to derive the mathematical properties of the first part of the decomposition process. Second, we wish to leave open the possibility of applying the skew representation of Chapter 6 to the partial skew tableaux produced by an ordered decomposition.

Note that in addition to encompassing the removal of nodes as per McAven et al., this definition encompasses the removal of nodes by Robinson [5] in Chapter 6. To do this, we simply choose sets of nodes in the ordered decomposition which correspond to the removal of an outermost diagonal strip tableau, followed by a next outermost diagonal strip tableau, and so on.

Recall from Chapter 2 that a composition of n is a partition of n without the weakly decreasing condition.

Definition 8.1 (†) *Let t be a partial skew tableau. An ordered decomposition of t into partial skew tableaux $(t_1, t_2, \cdots, t_k)$ is the k-tuple of partial skew tableaux produced from t by the ordered removal of sets of boxes.*

$$[(i_{k1}, j_{k1}), (i_{k2}, j_{k2}), \cdots, (i_{kl}, j_{kl_1})],$$

$$[(i_{(k-1)1}, j_{(k-1)1}), (i_{(k-1)2}, j_{(k-1)2}), \cdots, (i_{(k-1)m}, j_{(k-1)l_2})],$$

$$\vdots$$

$$[(i_{11}, j_{11}), (i_{12}, j_{12}), \cdots, (i_{1p}, j_{1l_j})].$$

We call the set of integers (l_i) the structure of the decomposition. Note that

$$\sum_i l_i = n,$$

so that the structure is a composition of n.

Example 8.1 *Let $\lambda = (6,3,1)$ be a partition of $n = 9$. Consider the partial skew tableau, t, with shape $\lambda/\nu = (5,3,1)/(2,1)$, given by*

$$\begin{array}{ccccc} & & \boxed{2} & \boxed{4} & \boxed{7} \\ & \boxed{3} & \boxed{5} & & \\ \boxed{8} & & & & \end{array} \quad .$$

This tableau represents the shape $(5,3,1)/(2,1)$. Then the following pair of tableaux is *an ordered decomposition of t into partial skew tableaux* (t_1, t_2) *with structure* $(3,3)$*:*

$$(t_1, t_2) = \left(\begin{array}{|c|c|c|} \hline 2 & 4 & 7 \\ \hline \end{array} \;,\; \begin{array}{ccc} & \boxed{3} & \boxed{5} \\ \boxed{8} & & \end{array} \right),$$

obtained by the removal of boxes

$$(t_{(2,2)}, t_{(2,3)}, t_{(3,1)}),$$

followed by the removal of boxes

$$(t_{(1,3)}, t_{(1,4)}, t_{(1,5)}).$$

Note that the order of removal within a set of boxes is immaterial, but the order of the sets of boxes is important. Thus the removal of the set of boxes

$$(t_{(1,3)}, t_{(1,4)}, t_{(1,5)}),$$

followed by the removal of boxes

$$(t_{(2,2)}, t_{(2,3)}, t_{(3,1)}),$$

produces the pair of tableaux

$$(t_1', t_2') = \left(\begin{array}{ccc} & \boxed{3} & \boxed{5} \\ \boxed{8} & & \end{array} \;,\; \begin{array}{|c|c|c|} \hline 2 & 4 & 7 \\ \hline \end{array} \right),$$

which is distinct from the pair (t_1, t_2). Note also that for convenience, we may treat the last set of boxes

$$(t_{(1,1)}, t_{(1,2)}, \cdots, t_{(1,p)}),$$

simply as those boxes remaining after the removal of all other sets of boxes.

Now consider an ordered decomposition of t with structure $(4,2)$, to produce the pair

$$(t_1'', t_2'') = \left(\begin{array}{ccc} \boxed{2} & \boxed{4} & \boxed{7} \\ \boxed{5} & & \end{array} , \begin{array}{cc} & \boxed{3} \\ \boxed{8} & \end{array} \right).$$

The first tableau is edge-connected and the second tableau is corner-connected. In general, the tableaux in the tuple may be edge-connected, corner-connected or unconnected. For example, consider an ordered decomposition of t with structure $(3,2,1)$, to produce the triplet

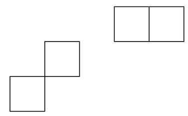
$$(t_1''', t_2''', t_3''') = \left(\begin{array}{cccc} & & \boxed{2} & \boxed{4} \\ \boxed{8} & & & \end{array} , \begin{array}{ccc} & & \boxed{7} \\ \boxed{3} & & \end{array} , \boxed{5} \right).$$

Both tableaux t_1''' and t_2''' are unconnected. Note also that a composition is an ordered sequence of non-negative integers, so that the composition $(3,1,2)$ with triplet

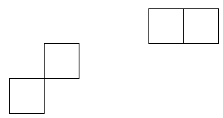
$$(t_1'''', t_2'''', t_3'''') = \left(\begin{array}{cccc} & & \boxed{2} & \boxed{4} \\ \boxed{8} & & & \end{array} , \boxed{5} , \begin{array}{ccc} & & \boxed{7} \\ \boxed{3} & & \end{array} \right).$$

has the same component tableaux as the previous triplet but in different order. The usually-decreasing order of a partition does not hold, so such a structure is permissable. It should also be pointed out that the original tableau may be edge-connected, corner-connected or unconnected.

In the case of an unconnected tableau, the number of nodes separating parts of the tableau is significant. Thus the shape $(5,2,1)/(3,1)$, shown below,



is distinct from the shape $(6,2,1)/(4,1)$, shown below



Tableaux having the same entries but different shapes are considered distinct. Thus we are considering an infinite number of shapes, and therefore an infinite number of tableaux.

It is useful to count the number of ordered decompositions of a tableau. The following theorem does this.

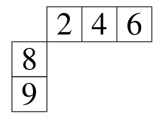
Theorem 8.1 (†) *Let t be a partial skew tableau with n boxes containing n distinct elements. Let $\mu = (l_1, l_2, \cdots, l_k)$ be a composition of n. The number of ordered decomposition of t into partial skew tableau having structure μ is given by*

$$\frac{n!}{l_1!l_2!\cdots l_k!}$$

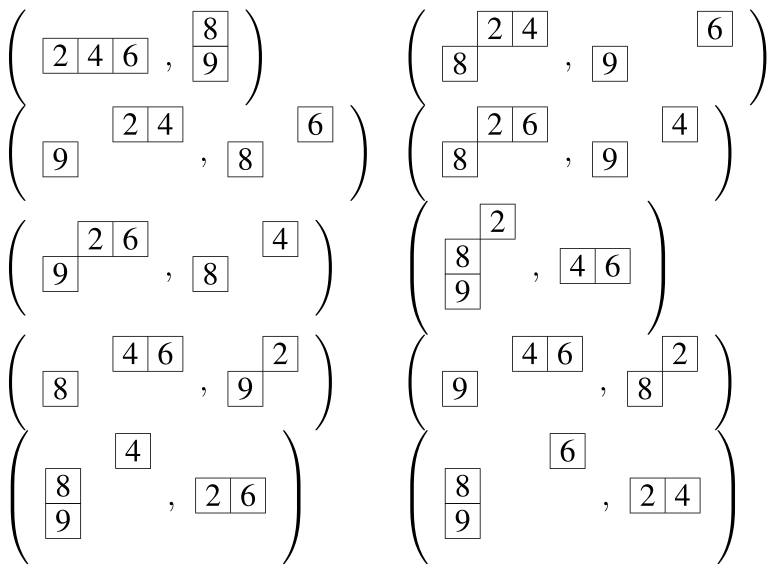
This is the number of orderings on n objects where the objects are indistinguishable.

Thus, $N = \frac{n!}{l_1!l_2!\cdots l_k!}$

Example 8.2 *Let t be the partial skew tableau*

$t =$  .

Let $\mu = (3,2)$. The complete list of ordered decompositions of t having structure μ is



Thus there are

$$\frac{5!}{3!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 10$$

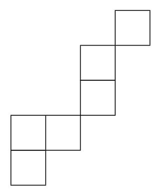
pairs of tableau, as predicted by Theorem 8.1.

Lemma 8.1 (†) *Let $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$ be the structure of an ordered decomposition of a partial tableau t. Let d be an ordered decomposition having structure μ. Then d defines a unique skew shape, which we shall denote by λ/ν.*

Proof. Consider the set of boxes

$$d_k = [(i_{k1}, j_{k1}), (i_{k2}, j_{k2}), \cdots (i_{kl}, j_{kl})]$$

removed from the partial tableau t. In this formulation, d_k denotes one set of nodes removed from a tableau in an ordered decomposition. Each of these boxes defines a unique node in the skew shape, say λ_k/ν_k, as shown below.



Similarly, the set of boxes d_{k-1}, defines a unique skew shape, λ_{k-1}/ν_{k-1}, and so on, until we reach the skew shape λ_1/ν_1. Each box is uniquely defined, so the complete set of boxes is the disjoint union of these k skew shapes, that is,

$$T = \biguplus_{i=1}^{k} \lambda_i/\nu_i.$$

This clearly defines a skew shape, which we shall denote by λ/ν.

Next we must show that λ/ν is unique. Suppose that there exists a second skew shape, λ'/ν', such that μ defines an ordered decomposition on λ'/ν'. Then one part of the structure must define two distinct skew shapes. Without loss of generality, we may assume that d_k defines distinct skew shapes, λ_k/ν_k and λ'_k/ν'_k. This requires that some box in d_k, say (i_{kl}, j_{kl}), defines two distinct locations, that is, (i_{kl}, j_{kl}) and (i'_{kl}, j'_{kl}). But clearly $i_{kl} = i'_{kl}$

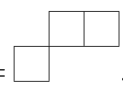
and $j_{kl} = j'_{kl}$. Hence t has a unique shape λ/ν.

□

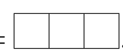
Example 8.3 *Let d be the ordered decomposition of Example 8.1, that is,*

$$d = \{[(1,3),(1,4),(1,5)],[(2,2)(2,3),(3,1)]\}.$$

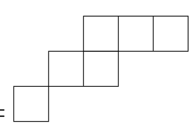
Then d_2 defines skew shape λ_2/ν_2, given by

$\lambda_2/\nu_2 =$  .

Also, d_1 defines skew shape λ_1/ν_1, given by

$\lambda_1/\nu_1 =$  .

The disjoint union of these skew shapes, λ/ν, is

$\lambda/\nu =$ 

This lemma establishes that there is one skew shape associated with an ordered decomposition. Further, this skew shape can be constructed wholly from the ordered decomposition.

We now establish that a composition may be the structure of several ordered decompositions.

Lemma 8.2 (†) *Let t be a partial skew shape tableau having n nodes and skew shape λ/ν. Let μ be a composition of n. Then there may be several ordered decompositions of t having structure μ.*

Proof. This is an immediate corollary of Theorem 8.1.

□

We have established in Lemma 8.1 and Lemma 8.2 that a composition defines several ordered decompositions. We now establish that there may be many such compositions associated with a skew shape.

Theorem 8.2 (†) *Let t be a partial skew tableau having n nodes and skew shape λ/ν. Then there may be many distinct compositions of n, each of which each define several distinct ordered decompositions of t.*

Proof. There are several partitions of n. A composition of n is a partition of n without the weakly decreasing condition. Hence there may be several compositions of n. From Lemma 8.2, each of these compositions may define several distinct ordered decompositions of t.

□

Example 8.4 *Let t be the tableau of Example 8.1, that is,*

$$t = \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & 2 & 4 & 7 \\ \cline{2-5} \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{2}{c}{} \\ \cline{1-3} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} .$$

Let $\mu = (3,3)$. Let d be the ordered decomposition of t with structure μ, given by

$$d = \{[(1,3),(1,4),(2,3)],[(1,5),(2,2),(3,1)]\}.$$

This decomposition gives the ordered pair of tableaux

$$(t_1,t_2) = \left(\begin{array}{cc} 2 & 4 \\ 5 & \end{array} \; , \; \begin{array}{ccccc} & & & & 7 \\ & 5 & & & \\ 8 & & & & \end{array} \right).$$

Now let d' be the ordered decomposition of t with structure μ, given by

$$d' = \{[(1,3),(2,2),(3,1)],[(1,4),(1,5),(2,3)]\}.$$

This decomposition gives the ordered pair of tableaux

$$(t'_1,t'_2) = \left(\begin{array}{ccc} & & 2 \\ & 3 & \\ 8 & & \end{array} \; , \; \begin{array}{ccc} & 4 & 7 \\ 5 & & \end{array} \right).$$

Let $\mu' = (2,2,2)$. Let d'' be the ordered decomposition of t with structure μ' given by

$$d'' = \{[(1,4),(15)],[(1,3),(2,3)],[(2,2),(3,1)]\}.$$

This decomposition gives the ordered triplet of tableaux

$$(t_1'', t_2'', t_3'') = \left(\begin{array}{|c|c|}\hline 4 & 7 \\ \hline \end{array} \ , \ \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \end{array} \ , \ \begin{array}{cc} & \fbox{3} \\ \fbox{8} & \end{array} \right).$$

Let d''' be the ordered decomposition of t with structure μ', given by

$$d''' = \{[(1,3),(1,4)],[(2,2),(2,3)],[(1,5),(3,1)]\}.$$

This decomposition gives the ordered triplet of tableaux

$$(t_1''', t_2''', t_3''') = \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array} \ , \ \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array} \ , \ \begin{array}{ccc} & & \fbox{7} \\ \fbox{8} & & \end{array} \right).$$

The tuples arising from these decompositions are all partial skew tableaus. Note that a partial skew tableau may be unconnected.

We are now able to establish the important result that an ordered decomposition is a mapping. This enables us to apply the properties of mappings to ordered decompositions.

Theorem 8.3 (†) *Let n be a positive integer. Let U be a set of n positive integers,*

$$U = \{u_1, u_2, \cdots, u_n\}.$$

Let T be the set of partial skew tableaux with elements drawn from U, that is,

$$T = \{t; u_i, \cdots, u_j |\ t \text{ is a partial skew tableau};$$

$$u_i, \cdots, u_j \in U; u_i, \cdots, u_j \text{ are nodes of } t\}.$$

Let T_n denote the subset of T with exactly n nodes, that is,

$$T_n = \{t | t \in T, t \text{ has exactly } n \text{ nodes}\}.$$

Let $\mu = \{\mu_1, \mu_2, \cdots, \mu_k\}$ be a composition of n.

Let α be an ordered decomposition of a tableau with structure μ, which defines a skew shape λ/ν. Let $T_{\lambda/\nu}$ be the subset of T_n with shape λ/ν. Then α is a mapping from the set of tableaux, $T_{\lambda/\nu}$, into the Cartesian product of tableaux, $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$.

That is, a unique partial skew tableau maps to one and only one k-tuple of partial skew tableaux under the action of α.

Proof. Let $t \in T_{\lambda/\nu}$. Since μ is a composition of n, it is a valid to form an ordered decomposition of t with structure μ. This holds for all such elements of $T_{\lambda/\nu}$. Hence the domain of the operation of ordered decomposition is the set of tableaux $T_{\lambda/\nu}$. The elements of the k-tuple of partial skew tableaux produced by the ordered decomposition are all elements of the set of tableaux, T. Hence the codomain of the operation is the Cartesian product

$$\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}.$$

To show that the ordered decomposition is indeed a mapping, we must show that it is well-defined. To do this, suppose that the tableau t decomposes into two distinct k-tuples of tableaux. These are

$$t_1 \times t_2 \times \cdots \times t_k, \quad \text{and} \quad t_1' \times t_2' \times \cdots \times t_k'.$$

Recall from our definition of an ordered decomposition that it requires the removal of boxes, first,

$$[(i_{k1}, j_{k1}), (i_{k2}, j_{k2}), \cdots, (i_{kl}, j_{kl})].$$

These boxes are distinct and unique. Let the boxes in t_k' be given by

$$t_{k1}, t_{k2}, \cdots, t_{kl},$$

and the boxes in t_{2k} be given by

$$t_{k1}', t_{k2}', \cdots, t_{kl}',$$

where t_{k1} and t_{k1}' are the boxes removed at (i_{k1}, j_{k1}),etc. Then as this box is unique, it follows that

$$t_{k1} = t_{k1}'.$$

Similarly $t_{k2} = t_{k2}', \cdots, t_{kl} = t_{kl}'$. Hence the tableau t_k is identical with the tableau t_k'. Similarly, $t_{(k-1)} = t_{(k-1)}', \cdots, t_1 = t_1'$. Therefore, these two k-tuples of tableaux are identical.

Hence the ordered decomposition is a mapping from $T_{\lambda/\nu}$ into

$$\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}.$$

□

Example 8.5 *Consider the tableau of Example 8.1, and the tableau pair* (t_1, t_2)*. In this example, U is the set of positive integers*

$$U = \{2, 3, 4, 5, 7, 8\}.$$

and $n = 6$*.* T_6 *is the set of tableaux with elements drawn from U. The tableau* $t \in T_n$ *is the tableau in the example with shape* $(5, 3, 1)/(2, 1)$*, and* $\mu = (3, 3)$*. T is the set of tableau with elements drawn from U. The ordered decomposition removing the set of boxes*

$$[(22), (23), (31)],$$

then

$$[(13), (14), (15)],$$

produces the tableau pair (t_1, t_2)*. This decomposition is a mapping from t to* (t_1, t_2)*, where* $t \in T_6$ *and* $(t_1, t_2) \in T \times T$*. In keeping with the usual notation of mappings, we may denote this mapping by a lowercase Greek letter, say* α*. Then*

$$\alpha(t) = (t_1, t_2).$$

Theorem 8.3 establishes that an ordered decomposition is a mapping. The following corollary shows that an ordered decomposition which acts on a subset of nodes is also a mapping.

Corollary 8.1 (†) *Let U, n and T be as in Theorem 8.3. Let m be a positive integer with $m < n$. Let U_m be a subset of U, given by*

$$U_m \subset U, |U_m| = m.$$

Let T_m be the subset of T with elements drawn from U_m. Let $w = (w_1, w_2, \cdots, w_l)$ be a composition of m. Let $t \in T_m$. Then an ordered decomposition of t into partial skew tableaux, with structure w, is a mapping from the set of tableaux $T_{\lambda/\nu}$ into the Cartesian product of tableaux

$$\underbrace{T \times T \times \cdots \times T}_{l \text{ terms}}.$$

Example 8.6 *Let U, n and T be as in Example 8.5. Let $m = 4$, and*

$$U_m = \{2, 3, 5, 8\}.$$

Let t be a tableau with shape $(4,2,1)/(2,1)$, where

$$t = \begin{array}{cccc} & & \boxed{2} & \boxed{5} \\ & \boxed{8} & & \\ \boxed{3} & & & \end{array} .$$

Let $\nu = (2,2)$ be a composition. We may write an ordered decomposition as

$$\beta(t) = (t_1, t_2)$$

where

$$(t_1, t_2) = \left(\begin{array}{ccc} & & \boxed{5} \\ \boxed{3} & & \end{array} , \begin{array}{cc} & \boxed{2} \\ \boxed{8} & \end{array} \right).$$

8.2 Properties of An Ordered Decompostion

Theorem 8.3 and Corollary 8.1 provide formal mathematical definitions of ordered decomposition of tableaux. Since an ordered decomposition of a tableau is a mapping, the mathematical properties of mappings may be applied to it. To facilitate this, we investigate the properties of this mapping. We first show that an ordered decomposition is an injective mapping.

Theorem 8.4 (†) *Let n, U, T, T_n, α and $T_{\lambda/\nu}$ be as in Theorem 8.3. Then α is a one-to-one mapping from the set $T_{\lambda/\nu}$ into the set*

$$\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}.$$

Proof. Let t be a tableau having shape λ/ν, that is, $t \in T_{\lambda/\nu}$. Let

$$\alpha(t) = (t_1, t_2, \cdots, t_k),$$

where t_1 has shape $\lambda_1/\nu_1, t_2$ has shape λ_2/ν_2, and so on. Suppose there exists another tableau $t' \in T_{\lambda/\nu}$ such that

$$\alpha(t') = \alpha(t) = (t_1, t_2, \cdots, t_k).$$

Consider the tableau t_k. It is formed by the removal of boxes

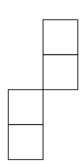
$$\{(i_{k1}, j_{k1}), (i_{k2}, j_{k2}), \cdots, (i_{kl}, j_{kl})\},$$

from both tableaux t and t'. Since the integers in these boxes must be identical in both t and t', it follows that these tableaux of shape λ_k/ν_k are identical in both t and t'. Similarly, the tableaux of shape λ_{k-1}/ν_{k-1} must be identical in both t and t', and so on. Therefore, the tableaux t and t' are identical since they are composed of identical component tableaux, that is, $t = t'$.

Hence, α is a one-to-one mapping. Thus we have the situation show in Figure 8.2.

□

Example 8.7 *Let* $U = \{1, 2, 3, 4\}$ *and* $n = 4$. *Let*

$\lambda/\nu =$  $= (2, 2, 1, 1)/(1, 1)$ *and* $\mu = (2, 2)$.

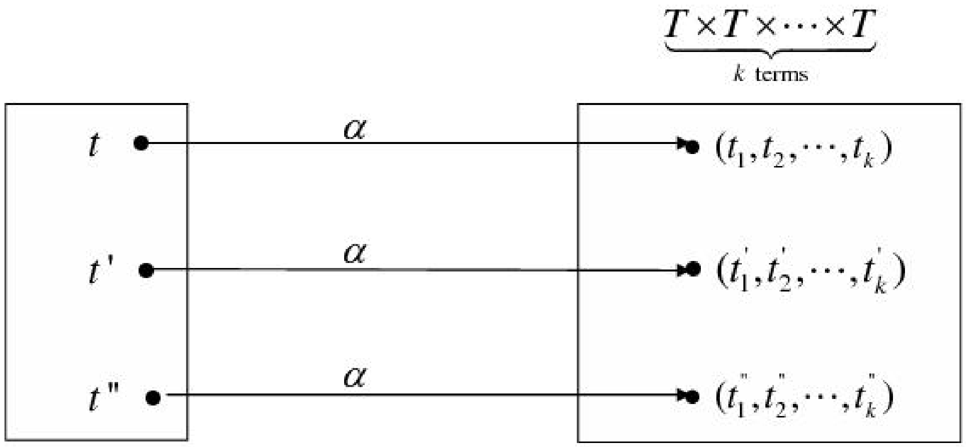


Figure 8.1: Mapping Theorem 8.4

Consider the ordered decomposition given by

$$\alpha = \{[(1,2),(2,2)],[(3,1),(4,1)]\}.$$

There are six partial skew tableaux having shape λ/ν. *The operation of the ordered* *decomposition α, is shown below.*

$$t_1 = \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{3} & \\ \boxed{4} & \end{array} \Rightarrow \left(\begin{array}{c} \boxed{1} \\ \boxed{2} \end{array}, \begin{array}{c} \boxed{3} \\ \boxed{4} \end{array} \right)$$

$$t_2 = \begin{array}{cc} & \boxed{1} \\ & \boxed{3} \\ \boxed{2} & \\ \boxed{4} & \end{array} \Rightarrow \left(\begin{array}{c} \boxed{1} \\ \boxed{3} \end{array}, \begin{array}{c} \boxed{2} \\ \boxed{4} \end{array} \right)$$

$$t_3 = \begin{array}{cc} & \boxed{1} \\ & \boxed{4} \\ \boxed{2} & \\ \boxed{3} & \end{array} \Rightarrow \left(\begin{array}{c} \boxed{1} \\ \boxed{4} \end{array}, \begin{array}{c} \boxed{2} \\ \boxed{3} \end{array} \right)$$

$$t_4 = \begin{array}{cc} & \boxed{2} \\ & \boxed{3} \\ \boxed{1} & \\ \boxed{4} & \end{array} \Rightarrow \left(\begin{array}{c} \boxed{2} \\ \boxed{3} \end{array}, \begin{array}{c} \boxed{1} \\ \boxed{4} \end{array} \right)$$

$$t_5 = \begin{array}{cc} & \boxed{2} \\ & \boxed{4} \\ \boxed{1} & \\ \boxed{3} & \end{array} \Rightarrow \left(\begin{array}{c} \boxed{2} \\ \boxed{4} \end{array}, \begin{array}{c} \boxed{1} \\ \boxed{3} \end{array} \right)$$

$$t_6 = \begin{array}{cc} & \boxed{3} \\ & \boxed{4} \\ \boxed{1} & \\ \boxed{2} & \end{array} \Rightarrow \left(\begin{array}{c} \boxed{3} \\ \boxed{4} \end{array}, \begin{array}{c} \boxed{1} \\ \boxed{2} \end{array} \right)$$

As predicted by Theorem 8.4, the ordered pairs of tableaux are distinct.

Theorem 8.4 is an important result in two ways.

First, it establishes that a set of distinct tableaux under the action of an ordered decomposition will each map to a distinct tuple of skew tableaux. This ensures the uniqueness of the tuples of skew tableaux produced by an ordered decomposition.

Second, it allows us to construct the original tableau from the tuple of skew tableaux, just by knowing the ordered decomposition. This is because an injective mapping is an invertible operation. Recall that in Chapter 4, we recorded the original tableau to construct an order on the original tableau. We have shown here that by knowing the ordered decomposition, we can construct the original tableau.

Next, we establish the cardinality of T_n, the set of partial skew tableaux having n nodes drawn from the set U. Recall that such tableaux may assume any skew shape.

Theorem 8.5 (†) *Let U, n, T and T_n be as in Theorem 8.3. T_n is the set of partial skew tableaux with exactly n nodes drawn from the set U. Then T_n defines an infinite number of skew shapes, that is, T_n is an infinite set, ie.*

$$|T_n| = \infty.$$

Proof. Suppose that T_n is finite, that is,

$$|T_n| = x,$$

for some positive integer x. Let t be an element of T_n with shape λ/ν, such that the rightmost box in row 1 of t is the rightmost box of any box in row 1 of any tableau in T_n. We may write this as the box $(1, j)$ of t. It is such that all boxes in row 1 of all tableaux in T_n occurs at positions $(1,k)$ for $1 \leq k \leq j$. Now consider the tableau t' which is identical to tableau t except that the box at $(1, j)$ is moved one node to the right, that is, at $(1, j+1)$. The tableau t' contains n nodes, so it is an element of T_n, that is, $t' \in T_n$. Hence $|T_n| = x+1$, contradicting the hypothesis. Hence T_n is an infinite set.

□

We wish to investigate the operation of an ordered decomposition on T_n, the set of partial skew tableaux having n nodes drawn from the set U. To do this, we define a relation on skew tableaux which have the same skew shape, as follows.

Definition 8.2 (†) *Let the symbol $\approxeq$ denote a relation between two tableaux t and t', meaning that tableau t' has the same skew shape as tableau t. That is, we write*

$$t \approxeq t',$$

if both t and t' have skew shape λ/ν.

Example 8.8 *Let tableaux t and t' be defined by*

$$t = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \boxed{4} \\ \boxed{3} & & \end{array} \quad \textit{and} \quad t' = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \boxed{3} \\ \boxed{4} & & \end{array} .$$

Both tableaux have skew shape $(3,3,1)/(2,1)$, so we write $t \approxeq t'$.

In order to show that the property of tableaux having the same skew shape partitions the set T_n, we first show that it is an equivalence relation.

Theorem 8.6 (†) *The relation $\approxeq$ between tableaux is an equivalence relation.*

Proof. Since a tableau has the same shape as itself, the relation is reflexive.

Next suppose that there exists a tableau t' such that $t \approxeq t'$. Since t' has the same shape as t, we may write $t' \approxeq t$. Hence the relation is symmetric.

Last suppose there exist a third tableau t'' such that $t' \approxeq t''$. Then as t as the same shape as t' and t' has the same shape as t'', it follows that t has the same shape as t''. Hence we may write $t \approxeq t''$. Therefore a relation is transitive. Hence the relation is an equivalence relation.

□

Example 8.9 *Let t and t' be as in Example 8.8, and let the tableau t'' be defined by*

$$t'' = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{3} & \boxed{4} \\ \boxed{2} & & \end{array} .$$

Since all three tableaux have skew shape $(3,3,1)/(2,1)$, we may write

$$t \approxeq t,$$

$$t \approxeq t',$$

$$t' \approxeq t,$$

$$t' \approxeq t'',$$

$$t \approxeq t''.$$

It follows that the relation of tableaux having the same skew shape partition the set of partial skew tableaux.

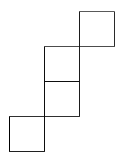
Corollary 8.2 (†) *Let ν, n and T_n be as in Theorem 8.3. Then the relation $\approxeq$ defined on the set T_n is a partition of T_n.*

Proof. Since the relation $\approxeq$ is an equivalence relation and an equivalence relation on a set defines a partition of a set, the relation $\approxeq$ defines a partition of the set T_n.

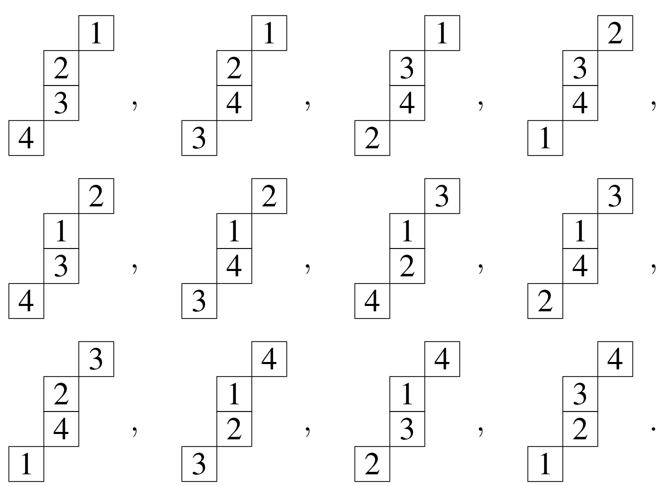
□

Example 8.10 *Let $V = \{1,2,3,4\}$ and $n = 4$. The set T_n is the set of partial skew tableaux having four boxes drawn from V. From Theorem 8.5, T_n is an infinite set. Let $\lambda/\nu = (2,2,1,1)/(1,1)$. The six tableaux having skew shape λ/ν were enumerated in Example 8.6.*

Now let $\lambda^*/\nu^*, = (3,2,2,1)/(2,1,1)$, that is,

$\lambda^*/\nu^* =$  .

There are twelve partial skew tableaux having shape λ^*/ν^*. These tableau are



Notice that these subsets are disjoint, that is,

$$T_{\lambda/\nu} \bigcap T_{\lambda^*/\nu^*}, = \phi.$$

Thus we have the situation shown in Figure 8.2.

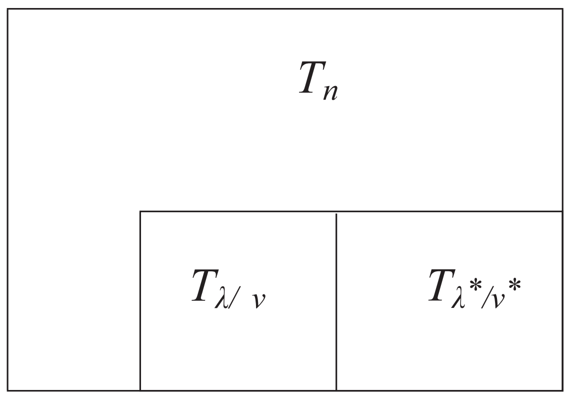


Figure 8.2: Partition of T_n

In Theorem 8.5, we showed that the set T_n is an infinite set. We now generalize this result to the case of partial skew tableaux having n nodes, regardless of the choice of integers used to fill the nodes.

Theorem 8.7 (†) *Let n be a positive integer. Then the number of skew shapes having n boxes is infinite.*

Proof. Suppose that the number of skew shapes having n boxes is finite, say x. Let λ/ν denote the shape having the rightmost box in row 1, say $(1, j)$. We may say that for all other such skew shapes, the rightmost box in row 1 is located at position $(1, k)$, where $1 \geq k \geq j$. Let λ^*/ν^* denote the skew shape with the box at $(1, j)$ moved one position to the right, that is, to position $(1, j+1)$. As this skew shape has n boxes, it is an element of the set of skew shapes having n boxes. Hence the number of skew shapes having n boxes is $x+1$. As this contradicts the hypothesis, the number of skew shapes having n boxes is infinite.

□

Corollary 8.3 (†) *The relation $\approxeq$ defined on the set T_n partitions T_n into an infinite number of infinite subsets, each being represented by a distinct skew shape.*

We also generalize Theorem 8.5 to the case of partial skew tableaux having any number of nodes.

Theorem 8.8 (†) *Let U and T be as in Theorem 8.3. Then T is an infinite set, that is, $|T| = \infty$.*

Proof. Suppose that T is a finite set consisting of x elements. Let $t \in T$ be the tableau containing the rightmost element in row 1, that is, t has a box at position $(1, j)$ and all other tableaux in T have their rightmost box at position $(1,k)$ for $1 \leq k \leq j$. Then let t' be the tableau t with node $(1, j)$ move right one position to $(1, j+1)$. Then $t' \in T$ since all its boxes contain elements from the set U. Hence $|T| = x+1$, contradicting the hypothesis. Hence T is an infinite set.

□

Corollary 8.4 (†) *The set $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$ is an infinite set.*

Definition 8.2 defined a relation on skew tableaux which have the same shape. The following definition generalizes this to the case of tuples of tableaux which pairwise have the same skew shape.

Definition 8.3 (†) *Let $\approxeq$ denote a relation on two k-tuples of tableaux with the meaning that $t_1 \times t_2 \times \cdots \times t_k \approxeq t'_1 \times t'_2 \times \cdots \times t'_k$ if corresponding tableaux in each tuple have the same skew shape, that is,*

$$t_j \approxeq t'{}_j \text{ for all } 1 \leq j \leq k.$$

This is obviously an extension of Definition 8.2 for tuples of tableaux. Obviously, if tuples have different numbers of tableaux, they are incompatible.

Example 8.11 *Let*

$$t = t_1 \times t_2 \times t_3 = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} , \begin{array}{cc} & \multicolumn{1}{|c|}{4} \\ \hline \multicolumn{1}{|c|}{5} & \\ \cline{1-1} \end{array} , \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right) \text{ and}$$

$$t' = t'_1 \times t'_2 \times t'_3 = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} , \begin{array}{cc} & \multicolumn{1}{|c|}{5} \\ \hline \multicolumn{1}{|c|}{6} & \\ \cline{1-1} \end{array} , \begin{array}{|c|} \hline 3 \\ \hline \end{array} \right).$$

Then we may write $t_1 \times t_2 \times t_3 \approxeq t' = t'_1 \times t'_2 \times t'_3$.

Theorem 8.6 stated that the relation defined in Definition 8.2 is an equivalence relation. The following theorem generalizes this result to the case of the relation defined on tuples of tableaux.

Theorem 8.9 (†) *The relation $\approxeq$ between k-tuples of tableaux is an equivalence relation.*

Proof. First we show that $\approxeq$ is reflexive. Let

$$t = t_1 \times t_2 \times \cdots \times t_k. \text{ by relexivity of } \approxeq \text{ on tableaux}$$

Clearly $t_j \approxeq t_j$ for all $1 \leq j \leq k$. Hence $t \approxeq t$.

Next we show that $\approxeq$ is symmetric. Let

$$t' = t'_1 \times t'_2 \times \cdots \times t'_k.$$

such that $t \approxeq t'$. Then by definition of $\approxeq$,

$$t_j \approxeq t'_j \text{ for all } 1 \leq j \leq k.$$

From Theorem 8.6, it follows that

$$t'_j \approxeq t_j \text{ for all } 1 \leq j \leq k, \text{ by symmetry of } \approxeq \text{ on tableaux}$$

Hence $t'_j \approxeq t_j$, so $\approxeq$ is symmetric.

Last we must show that $\approxeq$ is transitive. Let

$$t'' = t''_1 \times t''_2 \times \cdots \times t''_k,$$

such that

$$t' \approxeq t''.$$

Then from the definition of $\approxeq$,

$$t'_j \approxeq t''_j \text{ for all } 1 \leq j \leq k.$$

Hence

$$t_j \approxeq t''_j \text{ for all } 1 \leq j \leq k.$$

by transitivity of $\approxeq$ on tableaux. Hence $t \approxeq t''$, so $\approxeq$ is transitive.

$\square$

Example 8.12 *Let t and t' be as in Example 8.11, and let*

$$t'' = t''_1 \times t''_2 \times \cdots \times t''_3 = \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{c|c|} \cline{2-2} & 5 \\ \hline \multicolumn{1}{|c|}{6} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline \end{array} \right).$$

Then we may write

$$t \approxeq t' \approxeq t''.$$

It follows that the relation of Definition 8.3 defines a partition on tuples of tableaux.

Corollary 8.5 (†) *The relation $\approxeq$ describes a partition on the set $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$. This is a partition of an infinite set into an infinite number of infinite subsets.*

Thus we have the situation shown in Figure 8.3.

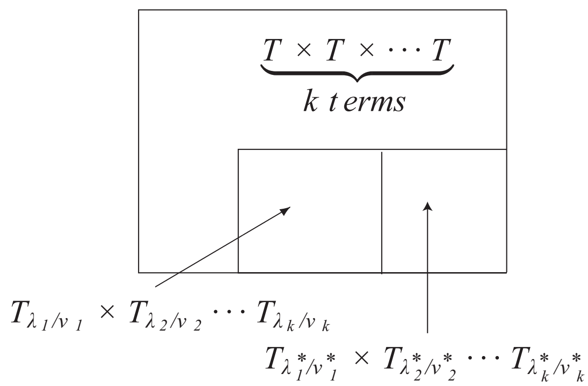


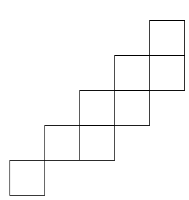
Figure 8.3: Partition of $T \times T \times \cdots \times T$

In order to study the properties of an ordered decomposition, we need to study the shapes produced by one or more decompositions. The following theorem assists in this study.

Given a composition which defines a structure of an ordered decomposition on a skew shape λ/ν, then there may be several such ordered decompositions which produce the tuple of skew shapes

$$\lambda_1/\nu_1 \times \lambda_2/\nu_2 \times \cdots \times \lambda_k/\nu_k.$$

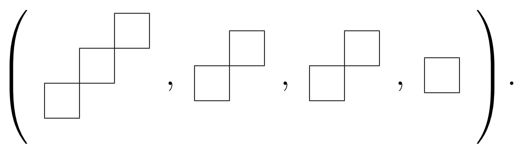
Example 8.13 *As an example consider the skew shape*

$\lambda/\nu =$  $= (5,5,4,3,1)/(4,3,2,1),$

and structure $\mu = (3,2,2,1)$. *The ordered decomposition, given by*

$$\alpha = \{[(1,5),(2,4),(3,3)],[(4,2),(5,1)],[(2,5),(3,4)][(4,3)]\},$$

produces the tuple of skew shapes



Also producing this tuple of skew shapes are the following decompositions -

$$\beta = \{[(1,5),(2,4),(3,3)],[(2,5),(3,4)],[(4,2),(5,1)],[(4,3)]\}$$

$$\gamma = \{[(1,5),(2,4),(3,3)],[(4,2),(5,1)],[(3,4),(4,3)],[(2,5)]\}$$

$$\xi = \{[(1,5),(2,4),(3,3)],[(3,4),(4,3)],[(4,2),(5,1)],[(2,5)]\}$$

$$\pi = \{[(3,3),(4,2),(5,1)],[(1,5),(2,4)],[(2,5),(3,4)],[(4,3)]\}$$

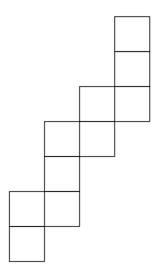
$$\Omega = \{[(3,3),(4,2),(5,1)],[(2,5),(3,4)],[(1,5),(2,4)],[(4,3)]\}$$

$$\Upsilon = \{[(3,3),(4,2),(5,1)],[(1,5),(2,4)],[(3,4),(4,3)],[(2,5)]\}$$

$$\Delta = \{[(3,3),(4,2),(5,1)],[(3,4),(4,3)],[(1,5),(2,4)],[(2,5)]\}$$

Example 8.14 *As another example consider the skew shape*

$\lambda^*/\nu^* =$

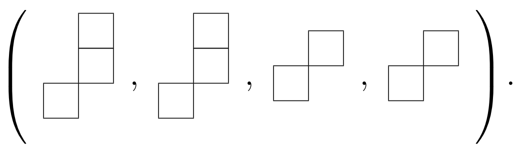


$= (4,4,4,3,2,2)/(3,3,2,1,1),$

and structure $\mu' = (3,3,2,2)$. *The ordered decomposition given by*

$$\alpha' = \{[(1,4),(2,4),(3,3)][(4,2),(5,2),(6,1)][(3,4),(4,3)][(6,2),(7,1)]\},$$

produces the tuple of skew shapes



Also producing this tuple of skew shapes are the following ordered decompositions :-

$$\beta' = \{[(4,2),(5,2),(6,1)][(1,4),(2,4),(3,3)][(3,4),(4,3)][(6,2),(7,1)]\},$$

$$\gamma' = \{[(1,4),(2,4),(3,3)][(4,2),(5,2),(6,1)][(6,2),(7,1)][(3,4),(4,3)],\}$$

$$\xi' = \{[(4,2),(5,2),(6,1)][(1,4),(2,4),(3,3)][(6,2),(7,1)][(3,4),(4,3)]\}.$$

8.3 Conclusion

We have derived a vigorous mathematical framework for the removal of nodes from a tableau. In the next chapter, we investigate the normalisation of the resulting tableau tuples using jeu de taquin.

Chapter 9

The Decomposition of Tableaux into Normal Tableaux

In the previous chapter, we developed a mathematical framework for the decomposition of a partial skew tableau into a tuple of partial skew tableau. In this chapter, we extend the mathematical framework to cover the normalisation of the tableau tuple using jeu de taquin.

9.1 Normalisation of Skew Tableau Tuples

The research of McAven et al. (see Chapter 7) was concerned with decomposing tableaux into tuples of normal tableaux. In order to further their research, we make the following definition.

Definition 9.1 (†) *Let t be a partial skew tableau. Then an ordered decomposition of t into partial normal tableaux, with structure μ, is an ordered decomposition of t into partial skew tableaux $(t_1, t_2, \cdots, t_k)$, followed by justification of each skew tableau, t_i, using jeu de taquin.*

This definition extends the definition of an ordered decomposition by providing for justification of the skew tableaux produced by an ordered decomposition, using jeu de taquin. This is the approach taken by McAven et al.

Example 9.1 *Let t be the tableau*

$$t = \begin{array}{ccccc} & & & & \boxed{2} \\ & & & \boxed{1} & \boxed{3} \\ & & & \boxed{4} & \boxed{10} \\ & & \boxed{5} & \boxed{11} & \\ & \boxed{7} & \boxed{13} & & \\ \boxed{6} & \boxed{15} & & & \end{array}$$

with skew shape $\lambda/\nu = (5,5,5,4,3,2)/(4,3,3,2,1)$. Let $\mu = (3,3,3,2)$ be the structure of an ordered decomposition of t into partial normal tableaux, defined by

$$\gamma = \{[(1,5),(2,4),(3,4)],[(4,3),(5,2),(6,1)],[(2,5),(3,5),(4,4)],[(5,3),(6,2)]\}.$$

Applying the ordered decomposition of t into partial skew tableaux yields the 4-tuple of skew tableaux

$$(t_1, t_2, t_3, t_4) = \left(\begin{array}{cc} & \boxed{2} \\ \boxed{1} & \\ \boxed{4} & \end{array} , \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} , \begin{array}{cc} & \boxed{3} \\ & \boxed{10} \\ \boxed{11} & \end{array} , \begin{array}{cc} & \boxed{11} \\ \boxed{15} & \end{array} \right).$$

Justification of these skew tableaux using jeu de taquin yields the 4-tuple of normal tableaux

$$(n_1, n_2, n_3, n_4) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|}\hline 3 \\ \hline 10 \\ \hline 11 \\ \hline \end{array} \; , \; \begin{array}{|c|}\hline 13 \\ \hline 15 \\ \hline \end{array} \right).$$

We seek to show that an ordered decomposition into normal tableaux is a composition of mappings. To do this, we show that justification of a tableau using jeu de taquin is a mapping.

Theorem 9.1 (†) *Let U, n and T be as in Theorem 8.3. Let N be the subset of T having normal shape. Then justification of elements of T using jeu de taquin is a mapping from T into N.*

Proof. Let $t \in T$ with skew shape λ/ν. Let $n \in N$ be the normal tableau resulting from justification of t using jeu de taquin, that is,

$$n = j(t).$$

Theorem 3.5 states that $j(t)$ is well-defined. Hence the operation $j(t)$ is a mapping from T into N. Theorem 3.4 further states that $j(t)$ is the Robinson-Schensted insertion tableau for the row word of t, π_i.

□

Example 9.2 *Consider the skew tableau t_1 from Example 9.1, that is,*

$$t_1 = \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad ,$$

with row word $\pi_{t_1} = 412$. Then

$$j(t) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

and this is the Robinson-Schensted insertion tableau for the permutation π_{t_1}.

It immediately follows that an ordered decomposition into normal tableaux is a composition of mappings.

Corollary 9.1 (†) *Let U, n, T, T_n, u and $T_{\lambda/\nu}$ be as in Theorem 8.3. Then an ordered decomposition, γ, of a tableau $t \in T_{\lambda/\nu}$ into partial normal tableaux is a mapping, α, from $T_{\lambda/\nu}$ into $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$ followed by a mapping, β, from $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$ into $\underbrace{N \times N \times \cdots \times N}_{k \text{ terms}}$. That is*

$$\gamma(t) = \beta(\alpha(t)) = \beta(t_1 \times t_2 \times \cdots \times t_k) = n_1 \times n_2 \times \cdots \times n_k.$$

Example 9.3 *Let γ be the ordered decomposition of Example 9.1. This mapping γ can be seen by inspection to comprise the composition of two mappings, α and β. The first mapping, α, maps t to a 4-tuple of partial skew tableaux, (t_1, t_2, t_3, t_4). The second mapping, β, maps this tuple to a 4-tuple of partial normal tableaux, (n_1, n_2, n_3, n_4).*

In the following pages, we wish to investigate the multiplicity case considered by McAven et al., that is, the case where several tuples of skew tableaux map to the same tuple of normal tableaux under jeu de taquin. The following lemma and theorems furnish the basis for the study of the multiplicity case.

Lemma 9.1 (†) *Let n be the normal tableau obtained by justification of the tableau t using jeu de taquin, that is,*

$$\begin{aligned} n &= j(t) \\ &= j^{C_m}, j^{C_{m-1}}, \cdots, j^{C_2} j^{C_1}(t), \end{aligned}$$

where C_i is the cell occupied by the i^{th} forward slide on t. Then there exists a sequence of backward slides which may be applied to n to yield t.

Proof. Let d_i be the cell vacated by the i^{th} forward slide on t. Let t_i be the tableau resulting from the i^{th} forward slide on t. Then we may write

$$\begin{aligned} t_0 &= t \\ t_1 &= j^{C_1}(t_0) \\ n &= t_m, \end{aligned}$$

with vacating cells

$$d_m d_{m-1} \cdots d_2 d_1.$$

However, we know from Theorem 3.4 that a slide is an invertible operation, that is,

$$j_{d_i} j^{C_i}(t_{i-1}) = t_{i-1}.$$

Specifically,

$$j_{d_m} j^{C_1}(t_{m-1}) = t_{m-1}.$$

We propose the sequence of backward slides to be

$$j_{d_1} j_{d_2} \cdots j_{d_{m-1}} j_{d_m}.$$

For we have that

$$
\begin{aligned}
& j_{d_1} j_{d_2} \cdots j_{d_{m-1}} j_{dm} j(t) \\
&= j_{d_1} j_{d_2} \cdots j_{d_{m-1}} j_{dm} j^{C_m}(t) \\
&= j_{d_1} j_{d_2} \cdots j_{d_{m-1}} (j_{dm} j^{C_m}(t)) \\
&= j_{d_1} j_{d_2} \cdots j_{d_{m-1}} (t_{m-1}) \\
&= j_{d_1} j_{d_2} \cdots j_{d_{m-1}} j^{C_{m-1}}(t_{m-2}) \\
&= j_{d_1} j_{d_2} \cdots (j_{d_{m-1}} j^{C_{m-1}}(t_{m-2})) \\
&= j_{d_1} j_{d_2} \cdots j_{d_{m-2}} (t_{m-2}) \\
&\;\;\vdots \\
&= j_{d_1}(t_1) \\
&= j_{d_1} j^{C_1}(t_0) \\
&= (j_{d_1} j^{C_1})(t_0) \\
&= t_0 \\
&= t.
\end{aligned}
$$

□

Example 9.4 *Let t be the second skew tableau from the 4-tuple of Example 9.3, that is,*

$$
t = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} .
$$

To justify t, we first perform a forward slide into cell $c_1 = (1,2)$, giving

$$
t_1 = j^{(1,2)}(t) = \begin{array}{cc} & \boxed{5} \\ & \boxed{7} \\ \boxed{6} & \end{array} .
$$

This vacates cell $d_1 = (1,3)$.

Next, we perform a forward slide into cell $c_2 = (2,1)$, *giving*

$$t_2 = j^{(2,1)}(t_1) = \begin{array}{cc} & \boxed{5} \\ \boxed{6} & \boxed{7} \end{array} .$$

This vacates cell $d_2 = (3,1)$. *We then perform a forward slide on cell* $c_3 = (1,1)$, *giving*

$$t_3 = j^{(1,1)}(t_2) = \begin{array}{cc} \boxed{5} & \\ \boxed{6} & \boxed{7} \end{array} .$$

This vacates cell $d_3 = (1,2)$. *Last, we perform a forward slide on cell* $c_4 = (1,2)$, *giving*

$$t_4 = j^{(1,2)}(t_3) = \begin{array}{cc} \boxed{5} & \boxed{7} \\ \boxed{6} & \end{array} .$$

This vacates cell $d_4 = (2,2)$.

From the preceding Lemma, we propose

$$j_{(1,3)} j_{(3,1)} j_{(4,2)} j_{(2,2)} t_4,$$

as the sequence of backward slides required to restore t *from* t_4. *We have*

$$j_{(2,2)} t_4 = \begin{array}{cc} \boxed{5} & \\ \boxed{6} & \boxed{7} \end{array} = t_3.$$

This vacates cell $c_4 = (1,2)$. Then, performing a backward slide into cell $(1,2)$ gives

$$j_{(1,2)}t_3 = \begin{array}{cc} & \boxed{5} \\ \boxed{6} & \boxed{7} \end{array} = t_2,$$

vacating cell $c_3 = (1,1)$. Next, performing a backward slide into cell $d_2 = (3,1)$ gives

$$j_{(3,1)}t_2 = \begin{array}{cc} & \boxed{5} \\ & \boxed{7} \\ \boxed{6} & \end{array} = t_1,$$

vacating cell $c_2 = (2,1)$. Last, we perform a backward slide into cell $d_1 = (1,3)$ giving

$$j_{(1,3)}t_1 = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} = t.$$

This vacates cell $c_1 = (1,2)$. Thus the sequence of backward slides restores tableau t from t_4.

Corollary 9.2 (†) *Let*

$$(n_1, n_2, \cdots, n_k) \in \underbrace{T \times T \times \cdots \times T}_{k \text{ terms}},$$

be the tuple of normal tableaux obtained by justification of the k-tuple of skew tableaux

$$(t_1, t_2, \cdots, t_k) \in \underbrace{T \times T \times \cdots \times T}_{k \text{ terms}},$$

using the k-tuple of forward slide sequence

$$(j_{b_1}, j_{b_2}, \cdots, j_{b_k}).$$

Then there exists a tuple of backward slide sequences which will restore $(t_1, t_2, \cdots, t_k)$ *from* $(n_1, n_2, \cdots, n_k)$.

Example 9.5 *Let* (t_1, t_2, t_3, t_4) *and* (n_1, n_2, n_3, n_4) *be as in Example 9.3. The 4-tuple of forward slide sequence is*

$$j_{f_1} = j^{(1,1)}, \text{ vacating cell } d_1 = (3,1);$$

$$j_{f_2} = j^{(1,2)} j^{(1,1)} j^{(2,1)} j^{(1,2)}, \text{ vacating cells}$$

$$d_1 = (1,3), \quad d_2 = (3,1),$$

$$d_3 = (1,2), \quad d_4 = (2,2);$$

$$j_{f_3} = j^{(1,1)} j^{(2,1)}, \text{ vacating cells}$$

$$d_1 = (2,2), \quad d_2 = (1,2);$$

$$j_{f_4} = j^{(1,1)}, \text{ vacating cells}$$

$$d_1 = (1,2).$$

It can be seen that the 4-tuple of backward slide sequences

$$j_{b_1} = j_{(3,1)}; \qquad j_{b_2} = j_{(1,3)} j_{(3,1)} j_{(1,2)} j_{(2,2)};$$

$$j_{b_3} = j_{(2,2)} j_{(1,2)}; \qquad j_{b_4} = j_{(1,2)},$$

will restore (t_1, t_2, t_3, t_4) *from* (n_1, n_2, n_3, n_4).

9.2 Properties of Normalisation of Tableau Tuples

We seek to show that normalization of a tuple of skew tableaux is a many-to-one mapping. To do this, we first show that normalization of a single tableau is a many-to-one mapping.

Theorem 9.2 (†) *Let n be a normal tableau. Then there may be several skew tableaux, not necessarily of the same shape, which map to n under jeu de taquin. That is, jeu de taquin is a many-to-one mapping.*

Proof. Since n has at least one node, it must have at least one outer corner, possibly more. Performing a backward slide into any of these outer corners produces a skew tableau, t, which maps to n under jeu de taquin. Now t must have at least one outer corner, possibly more. Performing a backward slide into any of these outer-corners produces a skew tableau t', which also maps to n under jeu de taquin, and so on. Thus there may be more than one skew tableau which maps to n under jeu de taquin.

□

Example 9.6 *Let n_1 be the normal tableau of Example 9.4, that is,*

$$n_1 = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 6 & \\ \cline{1-1} \end{array} \; .$$

This normal tableau was produced by justification of the skew tableau

$$t = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} \; .$$

It can be seen by inspection that the following skew tableaux also map to n under jeu de taquin;

$$t' = \begin{array}{cc} & 5 \\ & 7 \\ 6 & \end{array} ,$$

$$t'' = \begin{array}{cccc} & & & 5 \\ & 7 & & \\ 6 & & & \end{array} ,$$

$$t''' = \begin{array}{ccc} & & 5 \\ & 7 & \\ 6 & & \end{array} .$$

Corollary 9.3 (†) *Let*

$$(n_1, n_2, \cdots, n_k) \in \underbrace{N \times N \times \cdots \times N}_{k \text{ terms}},$$

be a k-tuple of normal tableaux. Then there may be more than one k-tuple of skew tableaux which map to $(n_1, n_2, \cdots, n_k)$ *under jeu de taquin of each tableau.*

Example 9.7 *Let* (n_1, n_2, n_3, n_4) *be the 4-tuple of normal tableau of Example 9.3, that is,*

$$(n_1, n_2, n_3, n_4) = \left(\begin{array}{cc} 1 & 2 \\ 4 & \end{array} , \begin{array}{cc} 5 & 7 \\ 6 & \end{array} , \begin{array}{c} 3 \\ 10 \\ 11 \end{array} , \begin{array}{c} 13 \\ 15 \end{array} \right).$$

This 4-tuple of normal tableaux was produced by performing jeu de taquin on the 4-tuple of skew tableaux.

$$(t_1, t_2, t_3, t_4) = \left(\begin{array}{cc} & 2 \\ 1 & \\ 4 & \end{array} , \begin{array}{ccc} & & 5 \\ & 7 & \\ 6 & & \end{array} , \begin{array}{cc} & 3 \\ & 10 \\ 11 & \end{array} , \begin{array}{cc} & 13 \\ 15 & \end{array} \right).$$

It can be seen from inspection that the following 4-tuple of skew tableaux also map to

(n_1, n_2, n_3, n_4) *under jeu de taquin of each tableau;*

$$(t'_1, t'_2, t'_3, t'_4) = \left(\begin{array}{ccc} & & 2 \\ & 1 & \\ 4 & & \end{array}, \begin{array}{ccc} & & 7 \\ & 5 & \\ 6 & & \end{array}, \begin{array}{ccc} & & 3 \\ & 10 & \\ 11 & & \end{array}, \begin{array}{cc} & 13 \\ 15 & \end{array} \right),$$

$$(t''_1, t''_2, t''_3, t''_4) = \left(\begin{array}{ccc} & & 1 \\ & 2 & \\ 4 & & \end{array}, \begin{array}{ccc} & & 5 \\ & 7 & \\ 6 & & \end{array}, \begin{array}{ccc} & & 10 \\ & 3 & \\ 11 & & \end{array}, \begin{array}{ccc} & & 13 \\ 15 & & \end{array} \right),$$

$$(t'''_1, t'''_2, t'''_3, t'''_4) = \left(\begin{array}{cc} & 2 \\ 1 & \\ 4 & \end{array}, \begin{array}{ccc} & & 7 \\ & 5 & \\ 6 & & \end{array}, \begin{array}{cc} & 3 \\ & 10 \\ 11 & \end{array}, \begin{array}{cc} & 13 \\ 15 & \end{array} \right).$$

Thus we have the situation shown in Figure 9.1. In this manner, we have a mathematical proof of an observation in section 5.2.

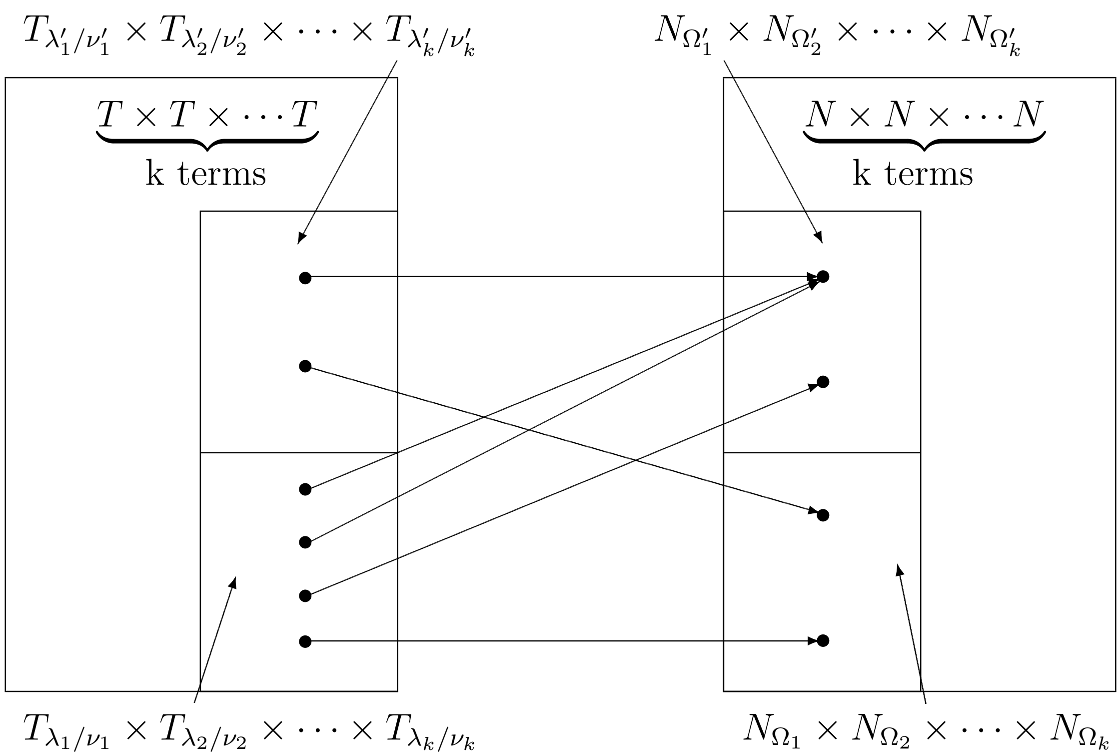


Figure 9.1: Mappings from $T \times T \times \cdots \times T$ to $N \times N \times \cdots \times N$

Corollary 9.4 (†) *There may be more than one partial skew tableau which produces the same tuple of normal tableaux under an ordered decomposition into partial normal tableaux.*

Example 9.8 *Let* (n_1, n_2, n_3, n_4) *be as in Example 9.7. Also let the 4-tuple of skew tableaux be as in that example. It can be seen that the skew tableau*

$$t = \begin{array}{|c|c|c|c|c|} \hline & & & & 2 \\ \hline & & & 1 & 3 \\ \hline & & & 4 & 10 \\ \hline & & 5 & 11 & \\ \hline & 7 & 13 & & \\ \hline 6 & 15 & & & \\ \hline \end{array},$$

maps to (t_1, t_2, t_3, t_4)*, which in turn maps to* (n_1, n_2, n_3, n_4)*. However the tableau*

$$t' = \begin{array}{|c|c|c|c|c|} \hline & & & & 2 \\ \hline & & & 1 & 3 \\ \hline & & & 4 & 10 \\ \hline & & 5 & 11 & \\ \hline & 7 & & & \\ \hline 6 & 13 & & & \\ \hline 15 & & & & \\ \hline \end{array}.$$

also maps to (t_1, t_2, t_3, t_4) *and then to* (n_1, n_2, n_3, n_4)*. However, this requires a different structure for the ordered decomposition.*

Now the skew tableau

$$t''' = \begin{array}{|c|c|c|c|c|} \hline & & & & 2 \\ \hline & & & 1 & 3 \\ \hline & & & 4 & 10 \\ \hline & & 7 & 11 & \\ \hline & 5 & 13 & & \\ \hline 6 & 15 & & & \\ \hline \end{array}.$$

maps to (t_1, t_2, t_3, t_4) *and then to* (n_1, n_2, n_3, n_4)*. This ordered decomposition has the same* *structure as that which acts on t to produce (n_1, n_2, n_3, n_4).*

Example 9.9 *Consider the decompositions of the tableaux of shape $\lambda = (3, 2, 1)$*

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array} ,$$

given in Example 5.5. The two tableaux

$$t_d = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} ,$$

and

$$t_g = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} ,$$

both map to the tableau pair

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 5 \\ \cline{1-1}\end{array} .$$

under an ordered decomposition into normal tableaux. The structures of these decompositions are identical. Thus they are the same decomposition.

The preceding results establish that an ordered decomposition into partial normal tableaux is a composition of a one-to-one mapping and a many-to-one mapping, as shown by the following theorem.

Theorem 9.3 (†) *An ordered decomposition of a partial skew tableau into partial normal tableaux is a composition of mappings. Specifically, it is a one-to-one mapping from a* *partial skew tableau to a k-tuple of partial skew tableaux, followed by a many-to-one mapping to a k-tuple of partial normal tableaux.*

Proof. Theorem 8.4 shows that the first part of the definition of an ordered decomposition of a partial skew tableau into partial normal tableaux is a one-to-one mapping from the set $T_{\lambda/\nu}$ into the set $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$.

Theorem 8.1 shows that the second part of the definition is a many-to-one mapping from the set $\underbrace{T \times T \times \cdots \times T}_{k \text{ terms}}$ into the set $\underbrace{N \times N \times \cdots \times N}_{k \text{ terms}}$.

Thus an ordered decomposition is a composition of mappings.

$\square$

Example 9.10 *Consider the partial skew tableau*

$$t = \begin{array}{ccccc} & & & & 2 \\ & & & 1 & 3 \\ & & & 4 & 10 \\ & & 5 & 11 & \\ & 7 & 13 & & \\ 6 & 15 & & & \end{array} ,$$

of Example 9.3. The ordered decomposition described first maps t to the 4-tuple of partial skew tableaux

$$(t_1, t_2, t_3, t_4) = \left(\begin{array}{cc} & 2 \\ 1 & \\ 4 & \end{array} , \begin{array}{ccc} & & 5 \\ & 7 & \\ 6 & & \end{array} , \begin{array}{cc} & 3 \\ & 10 \\ 11 & \end{array} , \begin{array}{cc} & 13 \\ 15 & \end{array} \right),$$

then maps this 4-tuple to the 4-tuple of normal tableaux

$$(n_1, n_2, n_3, n_4) = \left(\begin{ytableau} 1 & 2 \\ 4 \end{ytableau} \, , \, \begin{ytableau} 5 & 7 \\ 6 \end{ytableau} \, , \, \begin{ytableau} 3 \\ 10 \\ 11 \end{ytableau} \, , \, \begin{ytableau} 13 \\ 15 \end{ytableau} \right).$$

Example 9.11 *Consider the tableau*

$$t_d = \begin{ytableau} 1 & 2 & 4 \\ 3 & 6 \\ 5 \end{ytableau} \quad ,$$

of Example 9.9. The ordered decomposition described first maps t_d to the tableau pair

$$\left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \, , \, \begin{ytableau} \none & \none & 4 \\ \none & 6 \\ 5 \end{ytableau} \right),$$

then maps this tableau pair to the pair of normal tableaux

$$\left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \, , \, \begin{ytableau} 4 & 6 \\ 5 \end{ytableau} \right).$$

Thus in general we have the situation shown in Figure 9.2.

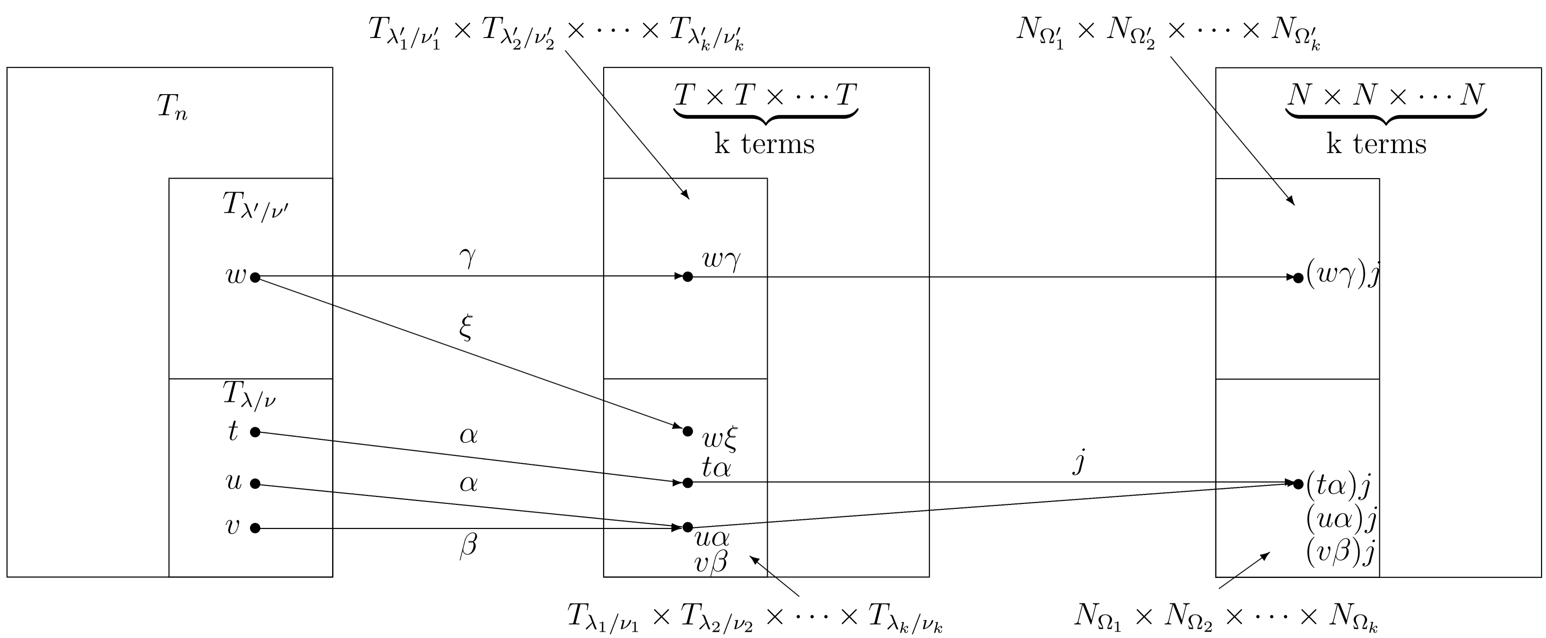


Figure 9.2: Mappings from T_n to $N \times N \times \cdots \times N$

In the research of McAven et al., we are particularly interested in the multiplicity case, that is, when several tableaux map to the same tuple of normal tableaux. To study this case further, we establish the following lemma.

Lemma 9.2 (†) *Let t_1 and t_2 be two partial skew tableaux which are justified to the same partial normal tableau, n_1, using jeu de taquin. Then t_2 may be obtained from t_1 by a sequence of slides, and vice-versa.*

Proof. Suppose that n_1 is obtained from t_1 by a sequence of forward slides, j_{f_1}, which comprises jeu de taquin. Then there is a corresponding sequence of backward slides, j_{b_1}, which restores t_1, from Lemma 9.1. Similarly, there exists sequences of slides, j_{f_2} and j_{b_2}, for t_2.

Now we may apply sequence j_{f_2} to t_2 to produce n_1, followed by j_{b_1} to produce t_1 from t_2. Similarly, we may apply j_{f_1} to t_1 to produce n_1, followed by j_{b_2} to produce t_2 from t_1. Hence, the tableaux t_1 and t_2 may be obtained from each other by sequences of slides.

□

Example 9.12 *Let n_1 be the normal tableau of Example 9.6, that is,*

$$n_1 = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 6 & \\ \cline{1-1}\end{array} \; .$$

Further, choose two tableaux from that example which justify to n_1 using jeu de taquin, *that is,*

$$t_1 = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} \quad ,$$

and

$$t_2 = \begin{array}{ccc} & & \boxed{7} \\ & \boxed{5} & \\ \boxed{6} & & \end{array} \quad .$$

The sequence of forward slides on t_1 corresponding to jeu de taquin is

$$t_1' = j_{2,1}(t_1) = \begin{array}{ccc} & & \boxed{5} \\ \boxed{6} & \boxed{7} & \end{array} \quad ;$$

$$t_1'' = j_{1,2}(t_1') = \begin{array}{cc} & \boxed{5} \\ \boxed{6} & \boxed{7} \end{array} \quad ;$$

$$n_1 = j_{1,1}(t_1'') = \begin{array}{cc} \boxed{5} & \boxed{7} \\ \boxed{6} & \end{array} \quad .$$

The corresponding sequence of backward slides is

$$t_1'' = j^{2,2}(n_1) = \begin{array}{cc} & \boxed{5} \\ \boxed{6} & \boxed{7} \end{array} \quad ;$$

$$t_1' = j^{1,3}(t_1'') = \begin{array}{ccc} & & \boxed{5} \\ \boxed{6} & \boxed{7} & \end{array} \quad ;$$

$$t_1 = j^{3,1}(t_1') = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} \quad .$$

Similarly, the sequence of forward slides on t_2, corresponding to jeu de taquin, is

$$t_2' = j_{2,1}(t_2) = \begin{array}{ccc} & & 7 \\ 5 & & \\ 6 & & \end{array} \; ;$$

$$t_2'' = j_{1,2}(t_2') = \begin{array}{cc} & 7 \\ 5 & \\ 6 & \end{array} \; ;$$

$$n_1 = j_{1,1}(t_2'') = \begin{array}{cc} 5 & 7 \\ 6 & \end{array} \; .$$

The corresponding sequence of backward slides is

$$t_2'' = j^{3,1}(n_1) = \begin{array}{cc} & 7 \\ 5 & \\ 6 & \end{array} \; ;$$

$$t_2' = j^{1,3}(t_2'') = \begin{array}{ccc} & & 7 \\ 5 & & \\ 6 & & \end{array} \; ;$$

$$t_2 = j^{2,2}(t_2') = \begin{array}{ccc} & & 7 \\ & 5 & \\ 6 & & \end{array} \; .$$

We may apply the sequence of slides to t_1

$$n_1 = j(t_1) = j_{1,1}(j_{1,2}(j_{2,1}(t_1))),$$

followed by the sequence of slides to n_1

$$t_2 = j^{2,2}(j^{1,3}(j^{3,1}(j(t_1)))),$$

to obtain tableau t_2 from tableau t_1. Similarly, we may obtain t_1 from t_2 by a sequence of *slides.*

Thus we have shown that two partial skew tableaux, t_1 and t_2, are equivalent, written $t_1 \approxeq t_2$, if they both justify to the same normal tableau, n_1 under the operation of jeu de taquin.

The preceding results now enable us to characterize the multiplicity case mathematically.

Theorem 9.4 (†) *Two partial skew tableaux, t_1 and t_2, are brought to the same normal tableau, n_1, under jeu de taquin, if and only if the are Knuth equivalent, written*

$$t_1 \overset{K}{\cong} t_2.$$

Proof. We established in Lemma 9.2 that

$$t_1 \cong t_2.$$

Theorem 3.5 shows that this condition is equivalent to the condition of Knuth equivalence, that is,

$$t_1 \overset{K}{\cong} t_2.$$

□

Example 9.13 *Let t_1 and t_2 be the two partial skew tableaux of the previous example, that* *is,*

$$t_1 = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{7} & \\ \boxed{6} & & \end{array} \quad \textit{and} \quad t_2 = \begin{array}{ccc} & & \boxed{7} \\ & \boxed{5} & \\ \boxed{6} & & \end{array} .$$

Then two tableaux are brought to the same normal tableau using jeu de taquin. Inspection shows that these two tableaux are Knuth equivalent, for their row words are

$$\pi_1 = 675 \quad \textit{and} \quad \pi_2 = 657.$$

It can be seen that

$$\pi_1 \overset{1}{\cong} \pi_2.$$

Corollary 9.5 *Consider the two k-tuples of partial skew tableaux,*

$$t = (t_1, t_2, \cdots, t_k) \quad \textit{and} \quad t' = (t'_1, t'_2, \cdots, t'_k).$$

The tuples, t and t', map to the same k-tuple of partial normal tableaux

$$n = (n_1, n_2, \cdots, n_k),$$

under jeu de taquin if and only if they are pairwise Knuth equivalent.

Example 9.14 *Let (t'_1, t'_2, t'_3, t'_4) and $(t''_1, t''_2, t''_3, t''_4)$ be the 4-tuples of tableaux of Example 9.7. It can be seen by inspection that*

$$t'_1 \overset{1}{\cong} t''_1, \quad t'_2 \overset{1}{\cong} t''_2, \quad t'_3 = t''_3, \quad \textit{and} \quad t'_4 = t''_4.$$

Thus these two 4-tuples of tableaux are pairwise Knuth equivalent.

Example 9.15 *Let t_d and t_g be as in Example 5.5, that is,*

$$t_d = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \textit{and} \quad t_g = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad ,$$

under the ordered decomposition into partial skew tableaux described by

$$\gamma = \{[(1,1),(1,2),(2,1)],[(1,3),(2,2),(3,1)]\}.$$

This gives the following 2-tuples of partial skew tableau

$$(t_{d_1}, t_{d_2}) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \; , \; \begin{array}{ccc} & & \multicolumn{1}{|c|}{4} \\ \cline{2-3} & \multicolumn{1}{|c|}{6} & \\ \cline{1-2} \multicolumn{1}{|c|}{5} & & \\ \cline{1-1}\end{array} \right),$$

$$(t_{g_1}, t_{g_2}) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \; , \; \begin{array}{ccc} & & \multicolumn{1}{|c|}{6} \\ \cline{2-3} & \multicolumn{1}{|c|}{4} & \\ \cline{1-2} \multicolumn{1}{|c|}{5} & & \\ \cline{1-1}\end{array} \right).$$

It can be seen by inspection that

$$t_{d_1} = t_{g_1}, \quad \textit{and} \quad t_{d_2} \overset{1}{\cong} t_{g_2}.$$

9.3 Conclusion

In the previous chapter we defined the decomposition of a partial skew tableau into a tuple of partial skew tableaux. In this chapter we have also defined the decomposition of a partial skew tableau into partial normal tableaux. The decompositions of a Young tableau may be considered to be special cases of these decompositions. Thus the decomposition given by McAven et al. is a special case of our more general mathematical framework. We have developed the set-theoretic and combinatorial aspects of these decompositions, in order to facilitate further research in this area.

Inspection shows that the decompositions in Chapter 6, which accounts for restricted representations, comprises sets of ordered decompositions of a tableau into skew tableau. This decomposition is taken from Robinson [5], who approaches the problem through the application of group character theory. The author believes that such an approach holds the most promise of finding the transition matrix between symmetric group bases. The material given in this chapter is intended to serve as an adjunct to such research. The development of the set-theoretic and combinatorial theory, in conjunction with the group character approach, could be the topic of further work.

Chapter 10

Conclusion and Further Research

We have introduced the representation theory of the symmetric group, with a view to deriving the transition matrix between symmetric group bases. We have given a survey of the research by Hamel et al. [7], and McAven et al. [8, 9] in this area.

The research by McAven et al. [7, 8, 9] hinges on the decomposition into subgroups of the symmetric group, and hence the Young tableaux in Young's representation of the symmetric group. We have initiated research into furnishing a formal mathematical framework for this. To do this we investigated the set-theoretic and combinatorial properties of the decomposition of tableaux. We have derived one significant result (Corollary 9.5). This result characterises tableaux which decompose to the same tuple of normal tableaux. In doing this we have established 4 definitions, proved 15 theorems, 4 lemmas and given 11 corollaries. We have also given 35 examples demonstrating the various theories, corollaries and lemmas given in this thesis.

We have assumed a knowledge of group character theory. Robinson [5] derives a decomposition in terms of character theory. This decomposition also has an analysis in terms of the decomposition of tableaux. Thus there is a link between the character-theoretic decomposition of Robinson [5] and the combinatorial decomposition derived in this thesis. It has been the author's intention to further investigate these two decompositions and the link between them. This would have been done by combining character theory with the set-theoretic and combinatorial principles developed in this thesis.

The decomposition by Robinson [5] hinges upon the removal of successively outermost diagonal strips from a tableau. This can be accommodated by an ordered decomposition by choosing sets of nodes in this fashion.

In Chapter 6, we introduced the skew representation. This could be applied to each tableau in the tuple of partial skew tableau produced by an ordered decomposition.

We have used Corollary 9.5 to characterize mathematically the multiplicity case investigated by McAven et al. Because tuples in the multiplicity case are pairwise Knuth equivalent, we have eliminated the need to justify the tuples of skew tableaux to determine which constitute the multiplicity case. Further, we can apply the skew representation to each tableau in the tuple of partial skew tableaux. This could be the subject of further research.

We have discussed polytabloids. The author spent a considerable amount of time investigating the application of polytabloids to finding the transition matrix. The nature of this investigation was computational rather than proof-theoretic. This investigation was not included in the thesis because it did not establish any new results. However, it did confirm an observation made by McAven et al. in their research. They observed that determining the matrix representation for the bridging transposition $(a, a+1)$ in the split basis adapted to $S_a \times S_b$ is the crux of the problem of finding the transition matrix. The author believes that this problem is best approached combinatorially, at least initially. Thus, to apply polytabloids to the problem requires the further development of the combinatorial approach undertaken in Chapter 8. This could be the subject of further research work. As we have characterised the multiplicity case mathematically, it may be possible to use this to devise on order on tuples of tableux based on their row words.

We have introduced symmetric functions. In this thesis, symmetric functions have not been applied to the problem of finding the transition matrix. However, the author believes that they may have proved useful in addressing the problem of the bridging transposition. This could be done by combining symmetric functions with the combinatorial approach taken in Chapter 8. Again, there arises the need to further develop the combinatorial theory of the decomposition of tableaux.

We have characterised the multiplicity case mathematically using combinatorial techniques. We have proposed the skew representation of Robinson[5]. as a means of resolving multiplicity separation. We have proved that multiplicity arises from normalisation of tuples of skew tableaux using je de taquin. The skew representation is applied to tuples of skew tableaux before normalisation, thus avoiding the problem of multiplicity.

Thus the focus of this thesis is addressing the issue of multiplicity separation. The other issue which must be addressed is that of finding a representation of the bridge transformation. McAven et al.[8],[9] propose the block selective conjecture as a means of resolving the bridging transposition. Some computational work undertaken by the author demonstrates that the skew representation, while addressing the issue of multiplicity, does not resolve the problem of the bridging transposition. Thus the problem of the bridge transposition could be the topic of future work. The papers by McAven et al.[8],[9] would be the starting point for such research.

The issue of subduction coefficients has not been considered in this thesis, but remains a possible focus for further research. The papers by Chilla[22],[23] have initiated research in this area. These papers could be the starting point for further research.

In this thesis, we have introduced the representation theory of the symmetric group. We have discussed polytabloids, symmetric functions and combinatorial methods. We have attempted to apply some of these techniques to finding the transition matrix between symmetric group bases. Specifically, we have studied combinatorial techniques. Further research work could bring the other methods to bear on the problem of finding the transition matrix. The combinatorial approach is common to these various techniques. Therefore, further research into combinatorial techniques is seen to be crucial in finding the transition matrix.

These approaches to the representation theory of the symmetric group could also be used to investigate the transition matrix between symmetric group bases.

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