EXCLUDING KURATOWSKI GRAPHS AND THEIR DUALS FROM BINARY MATROIDS

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Abstract. We consider various applications of our characterization of the internally 4-connected binary matroids with no $M(K_{3,3})$-minor. In particular, we characterize the internally 4-connected members of those classes of binary matroids produced by excluding any collection of cycle and bond matroids of $K_{3,3}$ and K_5, as long as that collection contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. We also present polynomial-time algorithms for deciding membership of these classes, where the input consists of a matrix with entries from GF(2). In addition we characterize the maximum-sized simple binary matroids with no $M(K_{3,3})$-minor, for any particular rank, and we show that a binary matroid with no $M(K_{3,3})$-minor has a critical exponent over GF(2) of at most four.

1. Introduction

In a previous article we proved the following theorem.

Theorem 1.1. [12, Theorem 1.1] *An internally* 4*-connected binary matroid M has no minor isomorphic to $M(K_{3,3})$ if and only if M is either:*

(i) cographic;
(ii) isomorphic to a triangular or triadic Möbius matroid; or,
(iii) isomorphic to one of 18 *sporadic matroids.*

The Möbius matroids are single-element extensions of bond matroids of Möbius ladders. We will describe them in Section 2.5. The 18 sporadic matroids appearing in Theorem 1.1 have ground sets of cardinality at most 21, and have rank at most 11. Matrix representations of the sporadic matroids appear in Appendix B of [12], and are also available from the website of the second author (`http://www.maths.uwa.edu.au/~gordon`).

This sequel explores various applications of Theorem 1.1. In Sections 1.1 to 1.4 we introduce these applications and state our main results. In Section 1.5 we collect some conjectures motivated by the results in this paper. Before proceeding, we note that the proof of Theorem 1.1 involves a considerable amount of computer checking. In this paper we also use a computer to check a handful of statements. For this purpose we use Macek, the software package developed by Petr Hliněný. Macek is freely available to download, along with supporting documentation, at the current website `http://www.fi.muni.cz/~hlineny/MACEK`.

1.1. **Other classes.** In Section 3 we characterize the internally 4-connected binary matroids with no minors in $\mathcal{M}$, where $\mathcal{M}$ is some subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ such that $\mathcal{M}$ contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. Thus we characterize the internally 4-connected members in twelve different families of binary matroids. Only the smallest of these classes has been characterized before [16].

1.2. **Polynomial-time algorithms.** In Section 4 we consider algorithmic consequences of Theorem 1.1. We present a polynomial-time algorithm for deciding if a binary matroid (represented by a matrix over GF(2)) has a minor in $\mathcal{M}$, where $\mathcal{M}$ is a subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$, and $\mathcal{M}$ contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. We also consider oracle algorithms.

Algorithms for binary matroids. Seymour's [18] famous decomposition theorem for regular matroids leads to a polynomial-time algorithm for deciding whether a matrix over GF(2) represents a regular matroid. In Section 4 we develop analogous algorithms for recognizing the twelve classes of binary matroids described in Section 3. The main theorem of Section 4 is the following result.

Theorem 1.2. *Suppose that $\mathcal{M}$ is a subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ such that $\mathcal{M}$ contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. There is a polynomial-time algorithm for solving the following problem: Given a matrix A over* GF(2)*, decide whether $M[A]$ has a minor in $\mathcal{M}$.*

The algorithms of Theorem 1.2 are necessarily more complicated than the algorithm for recognizing regular matroids, for the class of regular matroids is closed under 3-sums, while the classes in Theorem 1.2 are not. Much of Section 4.1 is spent developing more sophisticated ways of decomposing a binary matroid into its internally 4-connected components.

Oracle algorithms. Suppose that M is a matroid on the ground set E. When queried about a subset $X \subseteq E$ a *matroid oracle* returns in unit time some information about X. That information is typically the rank of X, or an answer to the question 'Is X independent?'. An oracle algorithm is *efficient* if the number of calls it makes to the oracle is bounded by some fixed polynomial function of $|E(M)|$, for any matroid M, and all additional computation can also be done in polynomial time.

Using Seymour's decomposition theorem, and techniques invented by Truemper [20], it is possible to construct an efficient oracle algorithm for deciding whether a matroid is regular (see [1, Section 7.4] and [21]). In contrast to this, the algorithms of Theorem 1.2 do not extend to efficient oracle algorithms, as we now discuss. An example of Seymour's [19] shows that there is no efficient oracle algorithm for deciding whether a matroid is binary. This same example shows that if $\mathcal{M}$ is any collection of bond and cycle matroids of Kuratowski graphs, then there can be no efficient oracle algorithm that decides if a matroid is binary with no minor in $\mathcal{M}$ (see Proposition 4.20). Thus we can expect no oracle analogue of Theorem 1.2.

On the other hand, the characterizations of Section 3 feature basic classes that are recognizable by efficient oracle algorithms. This reveals the curious fact that it is possible to have an efficient algorithm for deciding membership in a class of matroids when the input is guaranteed to be internally 4-connected, even if there is no efficient algorithm for deciding membership in the general case (see Propositions 4.20 and 4.21). We summarize this phenomenon in the following corollary.

Corollary 1.3. *Let $\mathcal{M}$ be a subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ such that $\mathcal{M}$ contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. There is no efficient oracle algorithm for deciding whether a matroid belongs to the class of binary matroids with no minor in $\mathcal{M}$. However, there is an efficient oracle algorithm for deciding whether an internally 4-connected matroid is a binary matroid with no minor in $\mathcal{M}$.*

Oracle algorithms are discussed more fully in Section 4.2.

1.3. **Maximum-sized binary matroids with no $M(K_{3,3})$.** In Section 5 we use Theorem 1.1 to determine the maximum size of a simple rank-r binary matroid with no $M(K_{3,3})$-minor. Moreover, we characterize the matroids that obtain this upper bound. This completely resolves a question studied by Kung [10]. He showed that a simple rank-r binary matroid M without an $M(K_{3,3})$-minor has at most $10r$ elements. Theorem 5.3 shows that, in fact, $|E(M)| \leq 14r/3 - \alpha(r)$, where $\alpha(r)$ assumes one of three values depending on the residue of r modulo 3. Moreover this bound is sharp. Any matroid meeting this bound can be obtained by starting with either PG(1, 2), PG(2, 2), or PG(3, 2), and then repeatedly adding copies of PG(3, 2) via parallel connections.

1.4. **Critical exponents.** If M is a matroid then its characteristic polynomial, $\chi(M; t)$ is a polynomial in the variable t. If M is loopless and representable over GF(q), then the "critical exponent" of M over q, denoted $c(M; q)$, is the smallest positive integer k such that $\chi(M; q^k) \neq 0$. In Section 6 we show that any loopless binary matroid with no $M(K_{3,3})$-minor has a critical exponent over GF(2) of at most four. Moreover, we characterize such matroids that have critical exponent equal to four: They are precisely those with a 3-connected component isomorphic to PG(3, 2) (see Theorem 6.2). This resolves a programme initiated by Kung [10], who showed that if M is a simple binary matroid with no $M(K_{3,3})$-minor, then $c(M; 2) \leq 10$.

1.5. **Conjectures on recognition problems.** In this section we speculate on the extent to which the algorithmic results of Section 4 are special cases of more general theorems. While this paper focuses on binary matroids, the conjectures are also interesting for other fields and we state some of them at that level of generality. The first two conjectures are originally due to Neil Robertson and Paul Seymour, although apparently they never stated them in print. They are discussed in [5] in the context of extending the Graph Minors project of Robertson and Seymour to matroids representable over finite fields.

Conjecture 1.4 (Well-Quasi-Ordering Conjecture)**.** Let GF(q) be a finite field. Then any infinite set of GF(q)-representable matroids contains two matroids, one of which is isomorphic to a minor of the other.

A positive answer to Conjecture 1.4 would imply that any minor-closed class of GF(q)-representable matroids has a finite number of GF(q)-representable excluded minors.

Conjecture 1.5. For any finite field GF(q) and any GF(q)-representable matroid N, there is a polynomial-time algorithm for solving the following problem: Given a matrix A over GF(q), decide whether $M[A]$ has a minor isomorphic to N.

We note that a positive answer to Conjecture 1.5 implies the famous result by Robertson and Seymour [17] that there is polynomial-time algorithm for detecting the presence of a fixed minor in a graph.

If Conjectures 1.4 and 1.5 were true, then the next conjecture would also hold.

Conjecture 1.6. For any finite field GF(q), and any minor-closed class $\mathcal{M}$ of GF(q)-representable matroids, there is a polynomial-time algorithm for solving the following problem: Given a matrix A over GF(q), decide whether $M[A]$ belongs to $\mathcal{M}$.

The results of Section 4 show that Conjecture 1.6 holds for the classes of binary matroids described in Section 3.

Just as intriguing is the possibility of extending the oracle-complexity results mentioned in Corollary 1.3. In what follows we restrict our attention to binary matroids. Seymour's [19] example shows that there is no efficient oracle algorithm for deciding whether a matroid is binary. Because of this example, one might expect that finding efficient oracle algorithms for recognizing classes of binary matroids is a hopeless task. But Corollary 1.3 makes it plausible that the difficulties may simply be due to degeneracies caused by low connectivity.

Conjecture 1.7. There is an efficient oracle algorithm for deciding if an internally 4-connected matroid is binary.

This is an ambitious conjecture. The next conjecture is somewhat more modest.

Conjecture 1.8. Let $\mathcal{M}$ be a proper minor-closed class of binary matroids. There is an efficient oracle algorithm for deciding whether an internally 4-connected matroid belongs to $\mathcal{M}$.

The matroids used by Seymour to construct his example are examples of "spikes". Spikes are a notorious source of difficulty in matroid theory. More generally, a *spike-like flower* of order n in a 3-connected matroid M is a partition $(P_1, \ldots, P_n)$ of the ground set of M such that, for every proper subset J of $\{1, \ldots, n\}$ the partition

$$(\bigcup_{j \in J} P_j,\, E(M) - \bigcup_{j \in J} P_j)$$

is an exact 3-separation of M; and, for all distinct i and j in $\{1, \ldots, n\}$ we have $r(P_i \cup P_j) = r(P_i) + r(P_j) - 1$. A rank-$n$ spike contains a spike-like flower of order n. We believe the existence of large spike-like flowers is at the heart of the difficulty of recognising binary matroids. This belief is encapsulated by the next conjecture, which is a strengthening of Conjecture 1.7.

Conjecture 1.9. Let k be a fixed positive integer. There is an efficient oracle algorithm for deciding if a 3-connected matroid with no spike-like flower of order k is a binary matroid.

Indeed the hypothesis of 3-connectivity in Conjecture 1.9 could be removed modulo the annoying technicalities of stating what it means for a more general matroid to have a spike-like flower. In fact, it is probably not difficult to prove that Conjecture 1.9 follows from Conjecture 1.7. Similar comments could be made about an analogous generalization of Conjecture 1.8.

2. Preliminaries

Our reference for fundamental notions of matroid theory is Oxley [15], and our notation follows that source, except that we denote the simple matroid associated with the matroid M by $\mathrm{si}(M)$. We assume that the ground set of $\mathrm{si}(M)$ is the set of parallel classes of M. If $\mathcal{M}$ is a collection of binary matroids, then $\mathcal{EX}(\mathcal{M})$ is the family of binary matroids with no minors in $\mathcal{M}$.

2.1. **Connectivity.** Let M be a matroid on the ground set E. The *connectivity function* of M, denoted by λ_M, takes any subset $X \subseteq E$ to $r_M(X) + r_M(E - X) - r(M)$. We use λ^*_M to denote the connectivity function of M^*. A partition (X_1, X_2) of E is a *k-separation* of M if $|X_1|, |X_2| \geq k$ and $\lambda_M(X_1) = \lambda_M(X_2) < k$. A k-separation (X_1, X_2) is *exact* if $\lambda_M(X_1) = k-1$. We say that M is *n-connected* if it has no k-separations such that $k < n$. In addition M is *internally 4-connected* if it is 3-connected and whenever (X_1, X_2) is a 3-separation, then $\min\{|X_1|, |X_2|\} = 3$.

The next result follows directly from a theorem of Oxley [14, Theorem 3.6].

Lemma 2.1. *Let T be a triangle of a 3-connected binary matroid M. If the rank and corank of M are at least three then M has an $M(K_4)$-minor in which T is a triangle.*

2.2. **Symmetric difference of matroids.** Suppose that M is a binary matroid. A *cycle* of M is a subset Z of $E(M)$ such that Z can be expressed as a (possibly empty) disjoint union of circuits of M. The symmetric difference of sets Z_1 and Z_2 is denoted by $Z_1 \triangle Z_2$. Binary matroids are characterized by the fact that the symmetric difference of any two cycles is another cycle.

Let M_1 and M_2 be two binary matroids on the ground sets E_1 and E_2 respectively. Let $\mathcal{Z}$ be the collection

$$\{Z_1 \triangle Z_2 \subseteq E_1 \triangle E_2 \mid Z_i \text{ is a cycle of } M_i,\ i = 1,\, 2\}.$$

Then $\mathcal{Z}$ is the collection of cycles of a binary matroid on the ground set $E_1 \triangle E_2$. We denote this matroid $M_1 \triangle M_2$.

Proposition 2.2. [18, (4.4)] *Suppose that M_1 and M_2 are binary matroids on the sets E_1 and E_2 respectively. If A and B are disjoint subsets of $E_1 - E_2$ then*

$$(M_1 \triangle M_2)/A \backslash B = (M_1/A \backslash B) \triangle M_2.$$

The next two results are straightforward to prove.

Proposition 2.3. *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively. Let $T = E_1 \cap E_2$, and assume that $M_1|T = M_2|T$. Then*

$$(M_1 \triangle M_2)|(E_1 - T) = M_1|(E_1 - T).$$

Proposition 2.4. *Suppose that M_1, M_2, and M_3 are binary matroids on the ground sets E_1, E_2, and E_3 respectively. Then*

$$(M_1 \triangle M_2) \triangle M_3 = M_1 \triangle (M_2 \triangle M_3).$$

Proposition 2.5. *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively. Assume $E_1 \subseteq E_2$. If $r(M_1) = 0$ then $M_1 \triangle M_2 = M_2/E_1$.*

Proof. Suppose that C is a circuit of M_2/E_1. Then $C \cup E_1$ contains a circuit C' of M_2 such that $C = C' - E_1$. Now $C' \cap E_1$ is a cycle of M_1 since it is a union of loops, so $C' - E_1 = C$ is a cycle of $M_1 \triangle M_2$, and hence contains a circuit of $M_1 \triangle M_2$. On the other hand, suppose that C is a circuit of $M_1 \triangle M_2$. Then $C = Z_1 \triangle Z_2$, where Z_1 is a cycle of M_1 and Z_2 is a cycle of M_2. There must be some circuit $C' \subseteq Z_2$ of M_2 such that $C' \cap (E_2 - E_1) \neq \varnothing$. Then C' contains a circuit of M_2/E_1. Thus every circuit of $M_1 \triangle M_2$ contains a circuit of M_2/E_1 and we are done. □

2.3. **The Δ-Y operation.** Suppose that M is a binary matroid and assume that $T = \{a_1, a_2, a_3\}$ is a triangle of M. Let N be an isomorphic copy of $M(K_4)$ such that $E(N) \cap E(M) = T$, where T is a triangle of N. Assume that $E(N) = T \cup \{a_1', a_2', a_3'\}$, where $(\{a_1', a_2', a_3'\} - a_i') \cup a_i$ is a triangle of N for $i = 1, 2, 3$. We say that $N \triangle M$ is produced from M by a $\Delta - Y$ *operation* on T, and we use $\Delta_T(M)$ to denote the resulting matroid. To ensure that M and $\Delta_T(M)$ have the same ground set we relabel a_i' with a_i in $\Delta_T(M)$, for $i = 1, 2, 3$.

Oxley, Semple, and Vertigan [13] generalize the Δ-Y operation using Brylawski's parallel connection [2]. It is easy to see that if T is coindependent in M, then the two definitions coincide. Many of the following results are identical to those in [13]. Because our definition of the Δ-Y operation is slightly different we provide some independent proofs.

Proposition 2.6. *Suppose that T is a coindependent triangle of the binary matroid M. Then T is a triad of $\Delta_T(M)$.*

Proof. Suppose $\Delta_T(M) = N \triangle M$. Recall that the elements of $E(N) - T$ are relabeled in $\Delta_T(M)$, so that $\Delta_T(M)$ and M have the same ground set. Now $E(N) - T$ is a cocircuit of N, and hence a cocircuit of N/T. Because T is coindependent in M it follows that T is a set of loops in $M/E(M) - T$. Propositions 2.2 and 2.5 imply that

$$N/T = N \triangle (M/E(M) - T) = (N \triangle M)/E(M) - T.$$

Therefore T is a cocircuit of $N \triangle M = \Delta_T(M)$. □

The next results follow easily from Propositions 2.3 and 2.4.

Proposition 2.7. *Suppose that T is a triangle of the binary matroid M. Then $\Delta_T(M)\backslash T = M\backslash T$.*

Proposition 2.8. *Suppose that M is a binary matroid and that T_1 and T_2 are disjoint triangles of M. Then*

$$\Delta_{T_1}(\Delta_{T_2}(M)) = \Delta_{T_2}(\Delta_{T_1}(M)).$$

Proposition 2.9. *Suppose that T is a triangle of the binary matroid M and that $a \in T$. Then $\Delta_T(M)/a = M\backslash a$.*

Proof. Suppose that $\Delta_T(M) = N \triangle M$. Let a' be the element of $E(N) - T$ that is relabeled with a. Proposition 2.2 says that $(N \triangle M)/a' = (N/a') \triangle M$. Note that N/a' consists of the triangle T with parallel elements added to both members of $T - a$. Suppose that $\{b, b'\}$ is a parallel pair of N/a', where $b \in T$. Let M^+ be obtained from M by adding x in parallel to b. Then $\{b, x\}$ is a cycle of M^+, so $\{b', x\}$ is a cycle of $(N/a') \triangle M^+$. Hence either $\{b', x\}$ is a parallel pair, or both b' and x are loops in $(N/a') \triangle M^+$. In either case Proposition 2.2 implies that

$$(N/a') \triangle M = (N/a') \triangle (M^+\backslash x) = ((N/a') \triangle M^+)\backslash x \cong ((N/a') \triangle M^+)\backslash b' = (N/a'\backslash b') \triangle M^+.$$

By using the same argument again we can show that $(N/a') \triangle M$ is isomorphic to the symmetric difference of $N|T$ and the matroid obtained from M by adding parallel elements to the members of $T - a$. Now the result follows easily from Proposition 2.3. □

Proposition 2.10. *Suppose that T is a triangle of the binary matroid M. Then $r(\Delta_T(M)) = r(M) + 1$.*

Proof. Let a be an element of T. Since a is not a coloop in M, nor a loop in $\Delta_T(M)$ the result follows immediately from Proposition 2.9. □

2.4. **Matroid sums.** In this section we define matroid 1-, 2-, and 3-sums, following the route taken by Seymour [18]. Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively. If E_1 and E_2 are disjoint, and neither E_1 nor E_2 is empty, then $M_1 \triangle M_2$ is the 1-*sum* of M_1 and M_2, denoted $M_1 \oplus M_2$. If E_1 and E_2 meet in a single element p, where p is a loop or coloop in neither M_1 nor M_2, and $|E_1|, |E_2| \geq 3$, then $M_1 \triangle M_2$ is the 2-*sum* of M_1 and M_2, denoted $M_1 \oplus_2 M_2$. We say that p is the *basepoint* of the 2-sum. Finally, suppose that $E_1 \cap E_2 = T$ and assume that the following conditions hold:

(i) T is a triangle in both M_1 and M_2;
(ii) T contains a cocircuit in neither M_1 nor M_2; and,
(iii) $|E_1|, |E_2| \geq 7$.

In this case $M_1 \triangle M_2$ is the 3-*sum* of M_1 and M_2, denoted $M_1 \oplus_3 M_2$.

Next we list a number of results due to Seymour.

Proposition 2.11. [18, (2.6)] *Suppose that (X_1, X_2) is an exact 2-separation of the binary matroid M. Then there are binary matroids M_1 and M_2 on the ground sets $X_1 \cup p$ and $X_2 \cup p$, where $p \notin X_1 \cup X_2$, such that $M = M_1 \oplus_2 M_2$. Conversely, if $M = M_1 \oplus_2 M_2$ then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is a 2-separation of M, and M_1 and M_2 are isomorphic to minors of M.*

Proposition 2.12. [18, (2.9)] *Suppose that (X_1, X_2) is an exact 3-separation of the binary matroid M such that $\min\{|X_1|, |X_2|\} \geq 4$. Then there are binary matroids M_1 and M_2 on the ground sets $X_1 \cup T$ and $X_2 \cup T$ respectively, where T is disjoint from $X_1 \cup X_2$, such that $M = M_1 \oplus_3 M_2$. Conversely, if $M = M_1 \oplus_3 M_2$, then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is an exact 3-separation of M.*

Proposition 2.13. [18, (4.1)] *Suppose that the binary matroid M can be expressed as the 3-sum of M_1 and M_2. If M is 3-connected then M_1 and M_2 are isomorphic to minors of M.*

Proposition 2.14. [18, (4.3)] *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively. Suppose also that the 3-sum of M_1 and M_2 is defined and that $M_1 \oplus_3 M_2$ is 3-connected. If (X_1, X_2) is a 2-separation of M_1, then for some $i \in \{1, 2\}$ we have that $X_i = \{x, z\}$, where $x \in E_1 - E_2$ and $z \in E_1 \cap E_2$. Moreover x and z are parallel in M_1.*

Proposition 2.15. *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively, where $E_1 \cap E_2 = T$. Suppose also that the 3-sum of M_1 and M_2 is defined. Let $M = M_1 \oplus_3 M_2$. If M is 3-connected then $\Delta_T(M_1)$ and $\Delta_T(M_2)$ are isomorphic to minors of M.*

Proof. Suppose that $r(M_1) < 3$. Proposition 2.14 implies that M_1 can have no parallel class containing more than two elements. Thus $|E_1| \leq 6$, a contradiction, as $M_1 \oplus_3 M_2$ is defined. Therefore $r(M_1) \geq 3$.

Suppose that $r(E_1 - \mathrm{cl}_{M_1}(T)) < r(M_1)$. Since $r(M_1) \geq 3$, it follows that $E_1 - \mathrm{cl}_{M_1}(T)$ is non-empty. If $|E_1 - \mathrm{cl}_{M_1}(T)| = 1$ then M_1 contains a coloop, and it is easy to see that this implies that $M_1 \oplus_3 M_2$ has a coloop, a contradiction. Therefore $|E_1 - \mathrm{cl}_{M_1}(T)| \geq 2$, and $(\mathrm{cl}_{M_1}(T), E_1 - \mathrm{cl}_{M_1}(T))$ is a 2-separation of M_1. This contradicts Proposition 2.14.

Therefore there is a basis B of M_1 that avoids $\mathrm{cl}_{M_1}(T)$. Proposition 2.14 implies $\mathrm{si}(M_1)$ is 3-connected and that the only parallel pairs of M_1 are contained in $\mathrm{cl}_{M_1}(T)$. Recall that the ground set of $\mathrm{si}(M_1)$ is the set of parallel classes of M_1. Thus $\mathrm{cl}_{M_1}(T)$ is a triangle of $\mathrm{si}(M_1)$. Since B is a basis of $\mathrm{si}(M_1)$ and $\mathrm{cl}_{M_1}(T)$ avoids B it follows that the rank and corank of $\mathrm{si}(M_1)$ are both at least three. Now Lemma 2.1 implies that $\mathrm{si}(M_1)$ has a minor isomorphic to $M(K_4)$ in which $\mathrm{cl}_{M_1}(T)$ is a triangle. Let M' be a minor of M_1 such that $M' \cong M(K_4)$ and T is a triangle of M'. Proposition 2.2 says that $M' \triangle M_2$ is a minor of M. However $M' \triangle M_2$ is isomorphic to $\Delta_T(M_2)$. The same argument shows that $\Delta_T(M_1)$ is isomorphic to a minor of M. □

Proposition 2.16. *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2, where $E_1 \cap E_2 = T$ and $M_1 \oplus_3 M_2$ is defined. Let T_0 be a triangle of M_1 that is disjoint from T. Then $\Delta_{T_0}(M_1 \oplus_3 M_2) = \Delta_{T_0}(M_1) \oplus_3 M_2$.*

Proof. Note that T_0 is a triangle of $M_1 \oplus_3 M_2$ by Proposition 2.3. Thus $\Delta_{T_0}(M_1 \oplus_3 M_2)$ is defined. Moreover T is a triangle of $\Delta_{T_0}(M_1)$ for the same reason. Let $\Delta_{T_0}(M_1) = N \triangle M_1$, where $N \cong M(K_4)$. If T contains a cocircuit in $N \triangle M_1$ then it contains a cocircuit in $(N \triangle M_1)/(E(N) - T_0) = (N/(E(N) - T_0)) \triangle M_1$, which is equal to M_1/T_0 by Proposition 2.5. Thus T contains a cocircuit in M_1, a contradiction as $M_1 \oplus_3 M_2$ is defined. Finally we observe that $|E(\Delta_{T_0}(M_1))| \geq 7$. Therefore $\Delta_{T_0}(M_1) \oplus_3 M_2$ is defined and Proposition 2.4 says that it is equal to $\Delta_{T_0}(M_1 \oplus_3 M_2)$. □

It is well known that if M_0 is a 2- (respectively, 3-) connected matroid, and neither M_1 nor M_2 has an M_0-minor, then the 1- (respectively, 2-) sum of M_1 and M_2 has no M_0-minor. We would like an analogue of this fact for 3-sums, but unfortunately the strict analogue is false: For example, the binary matroid R_{12} used by Seymour in his decomposition theorem for regular matroids can be expressed as the 3-sum of $M(K_5 \backslash e)$ and $M^*(K_{3,3})$. Neither of these matroids has an $M(K_{3,3})$-minor, but R_{12} does. However, Lemma 2.19 contains a partial results in this direction. Before we can prove the lemma, we need two preliminary propositions.

Proposition 2.17. *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively. Suppose that $E_1 \cap E_2 = \{e, f, g\}$, where $\{e, f, g\}$ is a triangle in M_2, and e is a loop while $\{f, g\}$ is a parallel pair* *in M_1. Let B_2 be a basis of M_2 such that $B_2 \cap \{e, f, g\} = \{e\}$. Then $B_2 - e$ is a basis of $(M_1 \triangle M_2)|(E_2 - E_1)$.*

Proof. Let $T = \{e, f, g\}$. Suppose that $B_2 - e$ contains a circuit C of $M_1 \triangle M_2$. In this case $C = Z_1 \triangle Z_2$, where $Z_1 \subseteq T$ is a cycle of M_1 and $Z_2 \subseteq B_2 \cup T$ is a cycle of M_2. Suppose that Z_2 is the disjoint union of the circuits $C_1, \ldots, C_t$ of M_2. No circuit of M_2 can be contained in B_2, so $t \leq 2$ as $|T - B_2| = 2$. Suppose that $t = 1$. It cannot be the case that $f, g \in C_1$, for then the symmetric difference of C_1 and T would be a circuit contained in B_2. Assume that $f \in C_1$. Then $Z_2 \cap T = Z_1$ contains f, but not g. This is impossible as Z_1 is a cycle, and f and g are parallel in M_1. This contradiction shows that $t = 2$. Since neither C_1 nor C_2 can contain $\{f, g\}$ we will assume that $f \in C_1$ and $g \in C_2$. As neither C_1 nor C_2 is contained in T it follows that both meet $B_2 - e$.

Suppose that e is in neither C_1 nor C_2. Then $(C_1 - f) \cup \{e, g\}$ and C_2 are distinct circuits contained in $B_2 \cup g$, a contradiction as B_2 is a basis of M_2. Thus we will assume that $e \in C_1$ (the argument when $e \in C_2$ is identical). This implies that $(C_1 - \{e, f\}) \cup g$ is a circuit, so C_2 must be equal to $(C_1 - \{e, f\}) \cup g$. But this is a contradiction, as C_1 and C_2 are disjoint.

We have shown that $B_2 - e$ is independent in $M_1 \triangle M_2$. Let x be an element in $E_2 - (B_2 \cup T)$. Then $B_2 \cup x$ contains a circuit C in M_2. As B_2 meets T in e, and e is a loop in M_1 it is now easy to see that $C - e$ is a cycle in $M_1 \triangle M_2$. Thus $(B_2 - e) \cup x$ is dependent in $M_1 \triangle M_2$. This implies that $B_2 - e$ is a basis of $E_2 - T$ in $M_1 \triangle M_2$, so we are done. □

Proposition 2.18. *Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively. Suppose that $E_1 \cap E_2 = \{e, f, g\}$, where $\{e, f, g\}$ is a coindependent triangle in M_2, and e is a loop while $\{f, g\}$ is a parallel pair in M_1. Then the restriction of $M_1 \triangle M_2$ to $E_2 - E_1$ is equal to $M_2/e\backslash f\backslash g$.*

Proof. Let $T = \{e, f, g\}$. Let C be a circuit of $M_2/e\backslash f\backslash g$. There is a circuit $C' \subseteq C \cup e$ of M_2 such that $C' - e = C$. As e is a loop of M_1 it follows that $C' - e = C$ is a cycle of $M_1 \triangle M_2$, and hence contains a circuit of $M_1 \triangle M_2$.

On the other hand, suppose that I is an independent set of $M_2/e\backslash f\backslash g$. Then $I \cup e$ is independent in M_2, and as T is coindependent in M_2 there is a basis B_2 of M_2 such that $I \cup e \subseteq B_2$ and $f, g \notin B_2$. Proposition 2.17 shows that $B_2 - e$, and hence I, is independent in $M_1 \triangle M_2$.

Suppose that X is some subset of $E_2 - T$. The previous arguments show that if X is dependent in $M_2/e\backslash f\backslash g$ then it is dependent in $M_1 \triangle M_2$, and if it is independent in $M_2/e\backslash f\backslash g$, then it is independent in $M_1 \triangle M_2$. This completes the proof. □

Lemma 2.19. *Let M_0 be an internally 4-connected binary matroid such that $|E(M_0)| \geq 4$. Suppose that M_1 and M_2 are binary matroids on the ground sets E_1 and E_2 respectively and that neither M_1 nor M_2 has an M_0-minor.*

Assume that $E_1 \cap E_2 = T$ *where* T *is a triangle of both* M_1 *and* M_2 *and that the* 3*-sum of* M_1 *and* M_2 *is defined. If* $M_1 \oplus_3 M_2$ *has an* M_0*-minor then either* $\Delta_T(M_1)$ *or* $\Delta_T(M_2)$ *has an* M_0*-minor and* M_0 *contains at least one triad.*

Proof. The hypotheses of the lemma imply that T contains a cocircuit in neither M_1 nor M_2, and that $|E_1|, |E_2| \geq 7$. Let $M = M_1 \oplus_3 M_2$, so that $M = M_1 \triangle M_2$. Proposition 2.12 asserts that $(E_1 - T, E_2 - T)$ is an exact 3-separation of M. Since M_0 is internally 4-connected it follows that either $|E(M_0) \cap (E_1 - T)| \leq 3$ or $|E(M_0) \cap (E_2 - T)| \leq 3$. By relabeling if necessary we will assume the former.

Let (A, B) be a partition of $E_1 - (T \cup E(M_0))$ such that M_0 is a minor of $M/A \backslash B$. We can assume that A is independent in M. Note that as $|E_1 - T| \geq 4$ it follows that $A \cup B$ is non-empty. Let $N = M_1/A\backslash B$. Proposition 2.2 says that $M/A\backslash B = N \triangle M_2$.

Assume that $r_N(T) = 0$. It is easy to see that $(E(N) - T, E(M_2) - T)$ is a 1-separation of $M/A\backslash B$. As M_0 is internally 4-connected and $E(M_0) \cap (E_2 - T)$ is non-empty it follows that $E(N) - T = \varnothing$. Thus $E(N) = T$ and N consists of three loops. Proposition 2.5 implies that $N \triangle M_2 = M_2/T$, and as M_0 is a minor of $M/A\backslash B = N \triangle M_2$ it follows that M_2 has an M_0-minor, a contradiction.

Next we assume that $r_N(T) = 1$. Since A is independent in M it must be independent in M_1. There is a circuit $C' \subseteq A \cup T$ of M_1 such that C' meets both A and T. Clearly C' cannot meet T in three elements as T is a triangle of M_1. Suppose that C' meets T in two elements. Then $C' \triangle T$ is a disjoint union of circuits of M_1. Thus there is a circuit $C \subseteq A \cup T$ such that C meets T in exactly one element. Let this element be e, and suppose that $T = \{e, f, g\}$. If $f \in \mathrm{cl}_{M_1}(A)$ then $T \subseteq \mathrm{cl}_{M_1}(A)$, which implies that $r_N(T) = 0$, contrary to hypothesis. Similarly $g \notin \mathrm{cl}_{M_1}(A)$, so e is a loop and $\{f, g\}$ is a parallel pair in N.

We will assume that $E(N) - T \neq \varnothing$. Assume that $\mathrm{cl}_N(E(N) - T)$ does not contain $\{f, g\}$. If C is a circuit of $N \triangle M_2$ that meets both $E(N) - T$ and $E_2 - T$ then there must be a cycle Z of N such that $Z - T = C \cap (E(N) - T)$. Let $C' \subseteq Z$ be a circuit of N such that $C' \cap (E(N) - T) \neq \varnothing$. It cannot be the case that $e \in C'$, as e is a loop of N. Similarly, C' cannot contain both f and g. Our assumption means that C' cannot contain precisely one of f and g. Therefore $C' \subseteq E(N) - T$. But this implies that C' contains a circuit of $N \triangle M_2$ that is properly contained in C, a contradiction. Therefore $\lambda_{N \triangle M_2}(E(N) - T) = 0$, and this means that $E(N) - T$ is empty, contrary to hypothesis. Henceforth we will assume that $\{f, g\} \subseteq \mathrm{cl}_N(E(N) - T)$.

Let B_1 be a basis of N such that $f \in B_1$. We show that there is no cycle Z of N such that $Z \subseteq B_1 \cup T$ and Z meets $B_1 - f$. Suppose that Z is such a cycle and let $C \subseteq Z$ be a circuit of N such that $C \cap (B_1 - f) \neq \varnothing$. As B_1 is independent in N it follows that C meets $T - f$. However C cannot contain e as it is a loop of N. Moreover C cannot contain $\{f, g\}$ as it is a parallel pair in N. Thus $C \cap T = \{g\}$. But then $(C - g) \cup f$ is a circuit of N contained in B_1, a contradiction.

Since T is a coindependent triangle of M_2 it follows that there is a basis B_2 of M_2 such that $B_2 \cap T = \{e\}$. The argument in the previous paragraph implies that if $(B_1 - f) \cup (B_2 - e)$ is dependent in $N \triangle M_2$, then $B_2 - e$ must contain a circuit C of $N \triangle M_2$. But Proposition 2.17 implies that $B_2 - e$ is independent in $N \triangle M_2$. Therefore $(B_1 - f) \cup (B_2 - e)$ is independent in $N \triangle M_2$, so

$$r(N) + r(M_2) - 2 \leq r(N \triangle M_2). \tag{1}$$

Let B be a basis of $N \triangle M_2$ restricted to $E(N) - T$. It cannot be the case that B contains a circuit C of N, for C would be a cycle in $N \triangle M_2$. Therefore B is independent in N, so

$$r_{N \triangle M_2}(E(N) - T) \leq r_N(E(N) - T) \leq r(N). \tag{2}$$

Proposition 2.17 shows that $B_2 - e$ is a basis of $E_2 - T$ in $N \triangle M_2$. Thus

$$r_{N \triangle M_2}(E_2 - T) = r(M_2) - 1. \tag{3}$$

By combining Equations (1), (2), and (3) we see that $\lambda_{N \triangle M_2}(E(N) - T) \leq 1$. Now we see that $E(N) - T$ contains exactly one element, x.

If x is not parallel to f and g in N then x is a loop or a coloop in N, and it is easy to see that it is therefore a loop or coloop in $N \triangle M_2$, which leads to a contradiction. Therefore $\{f, g, x\}$ is a parallel class of N. Suppose that M_2^+ is the matroid obtained from M_2 by adding an element x' in parallel to f. Now $\{f, x\}$ and $\{f, x'\}$ are cycles of N and M_2^+ respectively. Thus $\{x, x'\}$ is a cycle of $N \triangle M_2^+$ and therefore either $\{x, x'\}$ is a parallel pair, or both x and x' are loops of $N \triangle M_2^+$. In either case

$$N \triangle M_2 = N \triangle (M_2^+ \backslash x') = (N \triangle M_2^+) \backslash x' \cong (N \triangle M_2^+) \backslash x = (N \backslash x) \triangle M_2^+.$$

However Proposition 2.18 asserts that $(N \backslash x) \triangle M_2^+$ is equal to $M_2^+ / e \backslash f \backslash g$, which is isomorphic to $M_2 / e \backslash g$. Thus M_2 has an M_0-minor, a contradiction.

Therefore we suppose that $E(N) - T = \varnothing$. Proposition 2.18 tells us that $N \triangle M_2$ is equal to $M_2 / e \backslash f \backslash g$. As M_0 is a minor of $N \triangle M_2$ it follows that M_2 again has an M_0-minor, contrary to hypothesis.

We must now consider the case that $r_N(T) = 2$. Suppose that $E(N) - T$ contains a circuit of size at most two in N. Then $N \backslash T$ contains a circuit of size at most two. Since T is a triangle in both N and M_2 it follows from Proposition 2.3 that the restriction of $N \triangle M_2$ to $E(N) - T$ contains a circuit of size at most two. This implies that M_0 has a circuit with at most two elements, a contradiction.

Next we similarly assume that $E(N) - T$ contains a cocircuit of size at most two in N. Then N/T contains a cocircuit of size at most two. However, T is coindependent in M_2, which means that T comprises three loops in $M_2/(E(M_2) - T)$. Now Proposition 2.5 says that N/T is equal to

$$N \triangle (M_2/(E(M_2) - T)) = (N \triangle M_2)/(E(M_2) - T)).$$

Thus $E(N) - T$ contains a cocircuit of size at most two in $N \triangle M_2$. This implies that M_0 contains a cocircuit of size at most two, a contradiction.

We have shown that every circuit and cocircuit of N that is contained in $E(N) - T$ has size at least three. As $|E(N) - T| \leq 3$ it is now easy to see that either N is isomorphic to $M(K_4)$, or $r(N) = 2$ and N contains exactly three parallel classes, each one of size at most two. Suppose that N is isomorphic to $M(K_4)$. In this case $N \triangle M_2$ is equal to $\Delta_T(M_2)$, so $\Delta_T(M_2)$ has an M_0-minor. Moreover T is a triad of $\Delta_T(M_2)$ by Proposition 2.6. As $T \subseteq E(M_0)$ and M_0 has no cocircuits of size less than three it follows that M_0 has at least one triad, as desired.

Finally we assume that every element in $E(N) - T$ is parallel to an element of T in N, and that N contains no parallel class of more than two elements. By using the same arguments as before we see that we can replace every element in N that is parallel to an element in T with an element in M_2 that is parallel to the same member of T. Thus we can assume that $E(N) = T$. Now it follows easily from Proposition 2.3 that $N \triangle M_2$ is isomorphic to a minor of M_2. Thus M_2 has an M_0-minor, a contradiction. This completes the proof of the lemma. □

2.5. **Möbius matroids.** In this section we describe the Möbius matroids. The *cubic Möbius ladder* CM_{2n} is the cubic graph obtained from an even cycle with vertices $v_0, \ldots, v_{2n-1}$ by joining each vertex v_i to the antipodal vertex v_{i+n}. (Indices are to be read modulo $2n$.) The *quartic Möbius ladder* QM_{2n+1} is the quartic graph obtained from an odd cycle with vertices $v_0, \ldots, v_{2n}$ by joining each vertex v_i to the two antipodal vertices v_{i+n} and v_{i+n+1}. (In this case indices are read modulo $2n + 1$.) In either case, the edges of the cycle are known as *rim* edges, and the diagonal edges are known as *spokes.*

Triangular Möbius matroids. Let r be an integer exceeding two and let $\{e_1, \ldots, e_r\}$ be the standard basis in the vector space of dimension r over GF(2). For $1 \leq i \leq r-1$ let a_i be the sum of e_i and e_r, and for $1 \leq i \leq r-2$ let b_i be the sum of e_i and e_{i+1}. Let b_{r-1} be the sum of e_1, e_{r-1}, and e_r. The *rank-r triangular Möbius matroid*, denoted by Δ_r, is represented over GF(2) by the set $\{e_1, \ldots, e_r, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{r-1}\}$. Thus Δ_r has rank r and $|E(\Delta_r)| = 3r-2$. Deleting e_r from Δ_r produces a matroid isomorphic to the bond matroid of a cubic Möbius ladder. We say that $a_1, \ldots, a_{r-1}$ and $e_1, \ldots, e_{r-1}$ are the *rim elements* of Δ_r, and that $b_1, \ldots, b_{r-1}$ are the *spoke elements.* It is easy to see that if $r \geq 4$, then Δ_r has Δ_{r-1} as a minor.

Triadic Möbius matroids. Let $r \geq 4$ be an even integer, and again let $\{e_1, \ldots, e_r\}$ be the standard basis of the vector space over GF(2) of dimension r. For $1 \leq i \leq r-2$ let c_i be the sum of e_i, e_{i+1}, and e_r. Let c_{r-1} be the sum of e_1, e_{r-1}, and e_r. The *rank-r triadic Möbius matroid*, denoted by Υ_r, is represented over GF(2) by the set $\{e_1, \ldots, e_r, c_1, \ldots, c_{r-1}\}$. Thus Υ_r has rank r and $|E(\Upsilon_r)| = 2r-1$. If $r \geq 4$ is an even integer then $\Upsilon_r \backslash e_r$

is isomorphic to the bond matroid of a quartic Möbius ladder. We say that $e_1, \ldots, e_{r-1}$ are the *rim elements* of Υ_r and $c_1, \ldots, c_{r-1}$ are *spoke elements*. If r is an even integer greater than 4, then Υ_r has Υ_{r-2} as a minor.

3. Other classes of binary matroids

Recall that if $\mathcal{M}$ is a set of binary matroids, then $\mathcal{EX}(\mathcal{M})$ is the class of binary matroids that have no minors in $\mathcal{M}$. Let $\mathcal{M}$ be a subset of the collection

$$\{M(K_{3,3}),\, M^*(K_{3,3}),\, M(K_5),\, M^*(K_5)\}$$

with the property that $\mathcal{M}$ contains either $M(K_{3,3})$ or its dual. There are exactly twelve classes of binary matroids of the form $\mathcal{EX}(\mathcal{M})$. Theorem 1.1, and the famous graph-theoretical results of Hall [6] and Wagner [23], lead to characterizations of the internally 4-connected matroids in each of these classes. The first such characterization is given by Theorem 1.1.

Lemma 3.1. *The triangular and triadic Möbius matroids have no $M(K_5)$-minors.*

Proof. Lemma 3.8 of [12] states that the only internally 4-connected non-cographic minors of Möbius matroids are themselves Möbius matroids. Thus if a Möbius matroid had an $M(K_5)$-minor it would imply that $M(K_5)$ is a Möbius matroid. But the only rank-4 Möbius matroid with a ground set of size ten is Δ_4, and Δ_4 has only nine triangles. Thus no Möbius matroid has an $M(K_5)$-minor. □

The next result follows from Lemma 3.1 and a simple computer check.

Theorem 3.2. *An internally 4-connected binary matroid M has no $M(K_{3,3})$-minor and no $M(K_5)$-minor if and only if M is either:*

(i) cographic;
(ii) isomorphic to Δ_r for some integer $r \geq 3$ or to Υ_r for some even integer $r \geq 4$; or,
(iii) isomorphic to one of the following sporadic matroids: C_{11}, C_{12}, $M^a_{5,12}$, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, or $M_{11,21}$.

Wagner [23] characterized the graphs with no K_5-minor (see also [9, Theorem 1.6].) The following matroidal corollary of his theorem is well known, although its proof seems not to appear in the literature.

Lemma 3.3. *If M is an internally 4-connected cographic matroid with no minor isomorphic to $M^*(K_5)$ then either $M = M^*(G)$, where G is a planar graph, or M is isomorphic to one of $M^*(K_{3,3})$ or $M^*(CM_8)$.*

It is easy to check using a computer that Δ_6 has an $M^*(K_5)$-minor, and therefore Δ_r has an $M^*(K_5)$-minor for all $r \geq 6$. On the other hand Δ_r has no $M^*(K_5)$-minor if $r \in \{3, 4, 5\}$. Similarly Υ_r has an $M^*(K_5)$-minor if $r \geq 6$, but Υ_4 has no $M^*(K_5)$-minor. We note that $\Delta_3 \cong F_7$ and $\Upsilon_4 \cong F_7^*$. The next theorem follows from these facts, and by applying Theorem 1.1, Lemma 3.3, and some computer tests.

Theorem 3.4. *An internally* 4*-connected binary matroid* M *has no* $M(K_{3,3})$*-minor and no* $M^*(K_5)$*-minor if and only if* M *is either:*

(i) *planar graphic;*
(ii) *isomorphic to one of the cographic matroids* $M^*(K_{3,3})$ *or* $M^*(CM_8)$*;*
(iii) *isomorphic to one of the following Möbius matroids:* F_7, F_7^*, Δ_4, Δ_5*; or,*
(iv) *isomorphic to one of the* 18 *sporadic matroids of Theorem* 1.1*, other than* T_{12}*.*

A result due to Hall [6] implies that the only 3-connected cographic matroids with no $M^*(K_{3,3})$-minor are $M^*(K_5)$, and cycle matroids of planar graphs. The only Möbius matroids with no $M^*(K_{3,3})$-minor are $\Delta_3 \cong F_7$, $\Upsilon_4 \cong F_7^*$, and Υ_6. Corollary 2.15 of [12] says that the only internally 4-connected binary matroids that are non-cographic and have no minor isomorphic to either $M(K_{3,3})$ or Δ_4 are F_7, F_7^*, $M(K_5)$, $T_{12}\backslash e$, T_{12}/e, and T_{12}. A computer check reveals that none of these matroids has an $M^*(K_{3,3})$-minor. Both T_{12} and T_{12}/e are among the sporadic matroids of Theorem 1.1, while $T_{12}\backslash e \cong \Upsilon_6$. Since Δ_4 has an $M^*(K_{3,3})$-minor, the next result follows.

Theorem 3.5. *An internally* 4*-connected binary matroid* M *has no* $M(K_{3,3})$*-minor and no* $M^*(K_{3,3})$*-minor if and only if* M *is either:*

(i) *planar graphic;*
(ii) *isomorphic to the cographic matroid* $M^*(K_5)$*;*
(iii) *isomorphic to one of the following Möbius matroids:* F_7, F_7^*, *or* Υ_6*; or,*
(iv) *isomorphic to one of the following sporadic matroids:* $M(K_5)$, T_{12}/e, *or* T_{12}*.*

The next theorems are easy consequences of results stated above.

Theorem 3.6. *An internally* 4*-connected binary matroid* M *belongs to* $\mathcal{EX}(M(K_{3,3}), M(K_5), M^*(K_5))$ *if and only if* M *is either:*

(i) *planar graphic;*
(ii) *isomorphic to one of the cographic matroids* $M^*(K_{3,3})$ *or* $M^*(CM_8)$*;*
(iii) *isomorphic to one of the following Möbius matroids:* F_7, F_7^*, Δ_4, Δ_5*; or,*
(iv) *isomorphic to one of the following sporadic matroids:* C_{11}, C_{12}, $M_{5,12}^a$, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, *or* $M_{11,21}$*.*

Theorem 3.7. *An internally* 4*-connected binary matroid* M *belongs to* $\mathcal{EX}(M(K_{3,3}), M^*(K_{3,3}), M(K_5))$ *if and only if* M *is either:*

(i) *planar graphic;*
(ii) *isomorphic to the cographic matroid* $M^*(K_5)$*; or,*
(iii) *isomorphic to one of the following Möbius matroids:* F_7, F_7^*, *or* Υ_6*.*

Finally, we have the following characterization, which has already been proved by Qin and Zhou [16].

Theorem 3.8. *An internally* 4*-connected binary matroid* M *belongs to* $\mathcal{EX}(M(K_{3,3}),\ M^*(K_{3,3}),\ M(K_5), M^*(K_5))$ *if and only if* M *is either:*

(i) *planar graphic; or,*

(ii) *isomorphic to one of* F_7 *or* F_7^*.

By dualizing Theorems 1.1, 3.2, 3.4, 3.6, and 3.7, we obtain five additional characterizations of classes.

4. Polynomial-time algorithms

In this section we show that our structural characterizations lead to polynomial-time algorithms for deciding membership in each of the twelve classes of binary matroids described in Section 3; at least as long as the input consists of a matrix over GF(2). We will set aside consideration of oracle algorithms for the moment and return to them later.

4.1. **Algorithms for binary matroids.** For now we assume that a binary matroid on a ground set of size n is described by a matrix over GF(2) with n columns. We can assume that such a matrix has no more than n rows. The next result, due to Cunningham and Edmonds (see [1, Section 6.5.3]), relies upon the matroid intersection algorithm of Edmonds [4].

Proposition 4.1. *For each* $k \in \{1,\ 2\}$ *there is a polynomial-time algorithm that will either output a* k*-separation of a binary matroid* M*, or decide that* M *has no such separation. Moreover there is a polynomial-time algorithm that, for a binary matroid* M*, will either output a* 3*-separation* $(X_1,\ X_2)$ *of* M *such that* $|X_1|,\ |X_2| \geq 4$*, or decide that no such separation exists.*

If M is a rank-r binary matroid with no loops, then we can consider M as a multiset of points in the projective space $P = \mathrm{PG}(r-1,\ 2)$. If $X \subseteq E(M)$, then X is represented by a multiset of points in P, and we use $\mathrm{cl}_P(X)$ to denote the span of this set in P. The next result is well known, al beit difficult to find in the literature.

Proposition 4.2. *Let* M *be a binary matroid of rank* r*, and let* $P = \mathrm{PG}(r-1,\ 2)$*. Suppose that* $(X_1,\ X_2)$ *is an exact* k*-separation of* M *for some* $k \in \{1,\ 2,\ 3\}$ *with the property that if* $k = 3$ *then* $|X_1|,\ |X_2| \geq 4$ *and* $r_M(X_1),\ r_M(X_2) \geq 3$*. Let* $Z = \mathrm{cl}_P(X_1) \cap \mathrm{cl}_P(X_2)$*, and for* $i = 1,\ 2$ *let* M_i *be the binary matroid represented by the multiset* $X_i \cup Z$*. Then* $M \cong M_1 \oplus_k M_2$.

The next result is an easy consequence of Proposition 4.2.

Corollary 4.3. *There is a polynomial-time algorithm that, given a representation of a loopless binary matroid* M *and a partition* $(E(M_1) - E(M_2),\ E(M_2) - E(M_1))$ *of* $E(M)$*, where* M *is the* k*-sum of binary matroids* M_1 *and* M_2 *for some* $k \in \{1,\ 2,\ 3\}$*, will output representations of* M_1 *and* M_2.

The central idea of Seymour's algorithm for recognizing regular matroids, and of our algorithms, is that a binary matroid can be decomposed into internally 4-connected components. We now make this idea more formal.

Suppose that M is a binary matroid such that $\mathrm{si}(M)$ is 3-connected. Let $\mathcal{T}$ be a set of pairwise disjoint triangles in M. We recursively define a rooted tree, called a *decomposition tree* of $(M, \mathcal{T})$ (or just a decomposition tree of M), denoted by $\Phi(M, \mathcal{T})$. Each node is labeled with a matroid and a set of disjoint triangles of that matroid. Each node of $\Phi(M, \mathcal{T})$ has indegree one, apart from the root, which has indegree zero. Moreover each node has outdegree either zero or two. Those vertices with outdegree zero are called *leaves*.

If $\mathrm{si}(M)$ is internally 4-connected then $\Phi(M, \mathcal{T})$ comprises a single node: the root, which is labeled $(M, \mathcal{T})$. If $\mathrm{si}(M)$ is not internally 4-connected then there is an exact 3-separation (Y_1, Y_2) of $\mathrm{si}(M)$ such that $|Y_1|, |Y_2| \geq 4$. This naturally induces a separation of M, as follows: Recall that the ground set of $\mathrm{si}(M)$ is the set of parallel classes of M. Let (X_1, X_2) be the partition of $E(M)$ defined so that $x \in X_1$ if and only if $\mathrm{cl}_M(\{x\}) \in Y_1$. Thus (X_1, X_2) is an exact 3-separation of M, and $|X_1|, |X_2| \geq 4$. Let $r = r(M)$ and let $T = \mathrm{cl}_P(X_1) \cap \mathrm{cl}_P(X_2)$, where $P = \mathrm{PG}(r-1, 2)$. Thus if $M_i = P|(X_i \cup T)$ for $i = 1, 2$, then T is a triangle of both M_1 and M_2, and $M = M_1 \oplus_3 M_2$, by Proposition 4.2.

We want each of the triangles in $\mathcal{T}$ to be contained in either X_1 or X_2. If a triangle $T \in \mathcal{T}$ contains elements from both X_1 and X_2, then, up to relabeling, T contains exactly one element of X_2. We shift this element into X_1, and add a parallel element to take its place in X_2. More precisely: Suppose that $T_1, \ldots, T_p$ is the list of triangles in $\mathcal{T}$ that are contained in neither X_1 nor X_2. We make the following assignments. Let $M^{(0)} = M$. For $i = 1, 2$ let $M_i^{(0)} = M_i$ and let $X_i^{(0)} = X_i$. For $k = 1, \ldots, p$, let $\{i, j\} = \{1, 2\}$, and assume that exactly one element, e, of T_k is contained in $X_i^{(k-1)}$. We obtain $M^{(k)}$ by adding a new element e_k to $M^{(k-1)}$ so that it is parallel to e. We let $X_i^{(k)} = (X_i^{(k-1)} - e) \cup e_k$, and we let $X_j^{(k)} = X_j^{(k-1)} \cup e$. Since $e \in \mathrm{cl}_{M^{(k-1)}}(X_1^{(k-1)}) \cap \mathrm{cl}_{M^{(k-1)}}(X_2^{(k-1)})$ it follows that e must be parallel to an element t of T in $M_i^{(k-1)}$. We obtain $M_i^{(k)}$ from $M_i^{(k-1)}$ by relabeling the element e with e_k and we obtain $M_j^{(k)}$ from $M_j^{(k-1)}$ by adding e parallel to t. Note that $(X_1^{(k)}, X_2^{(k)})$ is an exact 3-separation of $M^{(k)}$, and that the number of rank-one flats in both $X_1^{(k)}$ and $X_2^{(k)}$ is at least four. It is easy to see that $M^{(k)} = M_1^{(k)} \oplus_3 M_2^{(k)}$.

Let $M^+ = M^{(p)}$, and for $i = 1, 2$ let $M_i^+ = M_i^{(p)}$. Thus $\mathrm{si}(M^+) = \mathrm{si}(M)$, and $M^+ = M_1^+ \oplus_3 M_2^+$. Moreover $\mathrm{si}(M_i^+) = \mathrm{si}(M_i)$ for $i = 1, 2$. Furthermore every triangle in $\mathcal{T}$ is contained in either $X_1^{(p)}$ or $X_2^{(p)}$. We let $(\mathcal{T}_1, \mathcal{T}_2)$ be the partition of $\mathcal{T}$ induced by $(X_1^{(p)}, X_2^{(p)})$.

Note that both $\mathrm{si}(M_1^+)$ and $\mathrm{si}(M_2^+)$ are 3-connected by Proposition 2.14. We recursively define $\Phi(M, \mathcal{T})$ to be the decomposition tree obtained by starting with the root, labeled $(M, \mathcal{T})$, adding two new vertices labeled by $(M_1^+, \mathcal{T}_1 \cup \{T\})$ and $(M_2^+, \mathcal{T}_2 \cup \{T\})$, adding arcs from the root to these two vertices, and then identifying the new vertices with the roots of $\Phi(M_1^+, \mathcal{T}_1 \cup \{T\})$ and $\Phi(M_2^+, \mathcal{T}_2 \cup \{T\})$ respectively.

Observe that, as M_1^+ and M_2^+ have strictly fewer rank-one flats than M, it must be the case that this recursive procedure will eventually terminate. Thus the leaves of $\Phi(M, \mathcal{T})$ are labeled with pairs $(M_1, \mathcal{T}_1), \ldots, (M_p, \mathcal{T}_p)$ such that $\mathrm{si}(M_i)$ is internally 4-connected for all $i \in \{1, \ldots, p\}$, and $\mathcal{T}_i$ is a set of pairwise disjoint triangles in M_i. Note that $\Phi(M, \mathcal{T})$ need not be unique: rather it depends upon our choice of 3-separations.

Proposition 4.4. *Suppose that M is a binary matroid such that $\mathrm{si}(M)$ is 3-connected and that $\mathcal{T}$ is a set of disjoint triangles of M. Let n be the number of rank-one flats of M. Then the number of leaves in $\Phi(M, \mathcal{T})$ is at most $\max\{1, n-6\}$.*

Proof. The proof is by induction on n. If $n \leq 7$ then $\mathrm{si}(M)$ is internally 4-connected, so $\Phi(M, \mathcal{T})$ has one leaf and we are done. Thus assume that $n > 7$ and that $\mathrm{si}(M)$ is not internally 4-connected. Suppose that the children of M in $\Phi(M, \mathcal{T})$ are labeled $(M_1^+, \mathcal{T}_1 \cup \{T\})$ and $(M_2^+, \mathcal{T}_2 \cup \{T\})$ respectively. For $i = 1, 2$ let the number of rank-one flats in M_i^+ be n_i. Then $n_1 + n_2 \leq n + 3$. Since $n_1, n_2 < n$ the inductive hypothesis implies that $\Phi(M_1^+, \mathcal{T}_1 \cup \{T\})$ and $\Phi(M_2^+, \mathcal{T}_2 \cup \{T\})$ have at most $\max\{1, n_1 - 6\}$ and $\max\{1, n_2 - 6\}$ leaves respectively. The result follows easily. □

Proposition 4.5. *There is a polynomial-time algorithm which, given a binary matroid M with the property that $\mathrm{si}(M)$ is 3-connected, and a set $\mathcal{T}$ of pairwise disjoint triangles of M, will compute a decomposition tree $\Phi(M, \mathcal{T})$.*

Proof. Let $n = E(M)$. We can assume that $n \geq 7$. Note that $|\mathcal{T}| \leq n/3$. Proposition 4.4 implies that the number of leaves in any decomposition tree of M is at most $n-6$. Thus $\Phi(M, \mathcal{T})$ has at most $n-7$ non-leaf vertices. It follows that if a node of $\Phi(M, \mathcal{T})$ is labeled with $(M_i, \mathcal{T}_i)$ then $|\mathcal{T}_i| \leq 4n/3 - 7$. Each triangle in $\mathcal{T}_i$ contributes at most one extra element to the matroids which label the children of $(M_i, \mathcal{T}_i)$. It follows that if $(M_j, \mathcal{T}_j)$ is a node label then M_j has at most $n + (4n/3 - 7)(n - 7)$ elements. Proposition 4.1 and Corollary 4.3 imply the existence of a polynomial-time algorithm which finds the two terms of each decomposition along a 3-sum. Suppose that this algorithm runs in time bounded by n^k for an n-element matroid, where k is a fixed constant. Clearly, given a representation of an n-element matroid, it is possible to construct a representation of the simplification of that matroid in time bounded by n^2. It follows that there is an algorithm that constructs $\Phi(M, \mathcal{T})$ in time bounded by

$$(n-7)(n + (4n/3 - 7)(n - 7))^{k+2}.$$ □

Proposition 4.6. *Let M_0 be a simple binary matroid. Suppose that M is a binary matroid such that* si(M) *is* 3*-connected and that $\mathcal{T}$ is a set of pairwise disjoint triangles of M. Let $\Phi(M, \mathcal{T})$ be a decomposition tree of M. If there is some leaf of $\Phi(M, \mathcal{T})$ labeled $(M_i, \mathcal{T}_i)$, where M_i has an M_0-minor, then M has an M_0-minor.*

Proof. The proof is by induction on the number of vertices in $\Phi(M, \mathcal{T})$. If $\Phi(M, \mathcal{T})$ has only one node then M_i must be equal to M and the result is trivial. Suppose that $\Phi(M, \mathcal{T})$ has more than one node, and suppose that the children of $(M, \mathcal{T})$ are labeled with $(M_1^+, \mathcal{T}_1 \cup \{T\})$ and $(M_2^+, \mathcal{T}_2 \cup \{T\})$. Without loss of generality we can assume that the leaf labeled by $(M_i, \mathcal{T}_i)$ belongs to $\Phi(M_1^+, \mathcal{T}_1 \cup \{T\})$. By induction M_1^+ has an M_0-minor. Proposition 2.13 says that $M^+ = M_1^+ \oplus_3 M_2^+$ has an M_0-minor. But $\mathrm{si}(M^+) = \mathrm{si}(M)$, and as M_0 is simple it follows that M has an M_0-minor. □

Proposition 4.7. *Suppose that M_0 is an internally* 4*-connected binary matroid such that $|E(M_0)| \geq 4$ and M_0 has no triads. Suppose also that M is a binary matroid such that* si(M) *is* 3*-connected, and that $\mathcal{T}$ is a collection of pairwise disjoint triangles of M. Let $\Phi(M, \mathcal{T})$ be a decomposition tree of M. If M has an M_0-minor then there is a leaf of $\Phi(M, \mathcal{T})$ labeled $(M_i, \mathcal{T}_i)$ such that M_i has an M_0-minor.*

Proof. The proof is by induction on the number of vertices in $\Phi(M, \mathcal{T})$. If $\Phi(M, \mathcal{T})$ has only one node the result is obvious. Suppose that $\Phi(M, \mathcal{T})$ has more than one node and assume that the children of $(M, \mathcal{T})$ are $(M_1^+, \mathcal{T}_1 \cup \{T\})$ and $(M_2^+, \mathcal{T}_2 \cup \{T\})$. If either M_1^+ or M_2^+ has an M_0-minor then the result follows by induction. Therefore we assume that neither M_1^+ nor M_2^+ has an M_0-minor. However, if $M^+ = M_1^+ \oplus_3 M_2^+$ then $\mathrm{si}(M^+) \cong M$. Therefore M^+ has an M_0-minor. Lemma 2.19 implies that M_0 contains at least one triad, a contradiction. □

Suppose that M is a binary matroid and that $\mathcal{T}$ is a set of pairwise disjoint triangles of M. We define $\Delta(M; \mathcal{T})$ to be the matroid produced by performing Δ-Y operations on each of the triangles in $\mathcal{T}$. Proposition 2.8 tells us that $\Delta(M; \mathcal{T})$ is well-defined. The next result follows from repeated application of Proposition 2.16.

Proposition 4.8. *Suppose that M_1 and M_2 are binary matroids and that the* 3*-sum of M_1 and M_2 along the triangle T is defined. For $i = 1, 2$ let $\mathcal{T}_i$ be a set of pairwise disjoint triangles of M_i that do not meet T. Then*

$$\Delta(M_1; \mathcal{T}_1) \oplus_3 \Delta(M_2; \mathcal{T}_2) = \Delta(M_1 \oplus_3 M_2; \mathcal{T}_1 \cup \mathcal{T}_2).$$

Proposition 4.9. *Let M_0 be a* 3*-connected binary matroid such that $|E(M_0)| \geq 4$ and M_0 has no triangles. Suppose that M is a binary matroid and that $\mathcal{T}$ is a collection of pairwise disjoint triangles of M. Suppose that the element e is in a parallel pair in M. If $\Delta(M; \mathcal{T})$ has an M_0-minor then $\Delta(M; \mathcal{T})\backslash e$ has an M_0-minor.*

Proof. The proof is by induction on the number of triangles in $\mathcal{T}$. If $\mathcal{T}$ is empty then the result is obvious, as M_0 has no parallel pairs. Suppose that $T_1, \ldots, T_p$ are the triangles in $\mathcal{T}$. Define $M^{(0)}$ to be M, and for $i = 1, \ldots, p$ define $M^{(i)}$ to be $\Delta_{T_i}(M^{(i-1)})$. Thus $M^{(p)} = \Delta(M;\, \mathcal{T})$. If e is in a parallel pair in $M^{(1)}$ then the result follows by the inductive hypothesis, as $M^{(p)} = \Delta(M^{(1)};\, \mathcal{T} - \{T_1\})$. Therefore we will assume that e is not in a parallel pair in $M^{(1)}$. Since e is in a parallel pair in $M^{(0)}$ we conclude that e is parallel to an element t of T_1 in $M^{(0)}$. If $p > 1$ then Proposition 2.7 implies that $\{e,\, t\}$ is a parallel pair in $\Delta_{T_2}(M)$. Using Proposition 2.8 we can apply the inductive hypothesis to $M^{(p)} = \Delta(\Delta_{T_2}(M);\, \mathcal{T} - \{T_2\})$ and conclude that the result holds. Therefore we will assume that $p = 1$.

Note that $\{e,\, t\}$ is a cycle of M. It is easy to see that $(T_1 - t) \cup e$ is a triangle of $\Delta_{T_1}(M)$. Let $T_1 - t = \{t_0,\, t_1\}$. Suppose that $\Delta_{T_1}(M)/t_0$ has an M_0-minor. As $\{e,\, t_1\}$ is a parallel pair in $\Delta_{T_1}(M)/t_0$ it follows that $\Delta_{T_1}(M)/t_0\backslash e$ has an M_0-minor, and we are done. Therefore we will assume that $\Delta_{T_1}(M)/t_0$ has no minor isomorphic to M_0, and (by the same argument), neither does $\Delta_{T_1}(M)/t_1$.

If t_0 is in a series pair of $\Delta_{T_1}(M)$ then $\Delta_{T_1}(M)/t_0$ must have an M_0-minor, contrary to our assumption. Therefore we assume that neither t_0 nor t_1 (by the same argument) is in a series pair in $\Delta_{T_1}(M)$. Propositions 2.7 and 2.10 imply that T_1 contains a cocircuit of $\Delta_{T_1}(M)$. Clearly neither t_0 nor t_1 is a coloop of $\Delta_{T_1}(M)$, so we conclude that either t is a coloop or T_1 is a triad in $\Delta_{T_1}(M)$. Suppose that the former holds. If there were a circuit of M which met T_1 in exactly t_0 or t_1 then we could find a circuit of $\Delta_{T_1}(M)$ which contained t. Therefore no such circuit exists. It follows that $\{t_0,\, t_1\}$ is a series pair in M. As $\{e,\, t_0,\, t_1\}$ is a triangle of $M\backslash t$ it follows that $\{t_0,\, t_1\}$ is also a series pair in $M\backslash t$, which is equal to $\Delta_{T_1}(M)/t$ by Proposition 2.9. Thus $\{t_0,\, t_1\}$ is a series pair of $\Delta_{T_1}(M)$, contrary to our conclusion. Therefore T_1 is a triad of $\Delta_{T_1}(M)$.

As M_0 contains no triangles we must delete an element of $(T_1 - t) \cup e$ from $\Delta_{T_1}(M)$ to obtain an M_0-minor. If this element is e then we are done, so assume that it is t_0 (the case when it is t_1 is identical). But $\{t,\, t_1\}$ is a series pair of $\Delta_{T_1}(M)\backslash t_0$, so $\Delta_{T_1}(M)\backslash t_0/t_1$, and hence $\Delta_{T_1}(M)/t_1$ has an M_0-minor, contrary to our earlier conclusion. This completes the proof. □

Lemma 4.10. *Suppose that M_0 is an internally 4-connected binary matroid such that $|E(M_0)| \geq 4$ and M_0 has no triangles. Suppose also that M is a binary matroid such that $\mathrm{si}(M)$ is 3-connected, and that $\mathcal{T}$ is a collection of pairwise disjoint triangles of M. Let $\Phi(M,\, \mathcal{T})$ be a decomposition tree of M. If M has an M_0-minor then there is a leaf of $\Phi(M,\, \mathcal{T})$ labeled $(M_i,\, \mathcal{T}_i)$ with the property that $\Delta(M_i;\, \mathcal{T}_i)$ has an M_0-minor.*

Proof. Let $\mathcal{T} = \{T_1, \ldots, T_p\}$. Define $M^{(0)}$ to be M, and for $i \in \{1, \ldots, p\}$ let $M^{(i)}$ be $\Delta_{T_i}(M^{(i-1)})$. Thus $M^{(p)} = \Delta(M;\, \mathcal{T})$. We start by showing that $\Delta(M;\, \mathcal{T})$ has an M_0-minor. If this is not the case there is some $i \in \{1, \ldots, p\}$ such that $M^{(i-1)}$ has an M_0-minor but $M^{(i)}$ does not. Since T_i is a triangle of $M^{(i-1)}$ and M_0 has no triangles it follows that there is an element $a \in T_i$ such that $M^{(i-1)}\backslash a$ has an M_0-minor. But Proposition 2.9 says that

$$M^{(i)}/a = \Delta_{T_i}(M^{(i-1)})/a = M^{(i-1)}\backslash a.$$

Therefore $M^{(i)}$ has an M_0-minor, a contradiction.

Suppose that the lemma is false. The argument in the previous paragraph shows that there is at least one node $(M_i,\, \mathcal{T}_i)$ such that $\Delta(M_i;\, \mathcal{T}_i)$ has an M_0-minor. Suppose that $(M_i,\, \mathcal{T}_i)$ has been chosen so that if $(M_j,\, \mathcal{T}_j)$ is a descendant of $(M_i,\, \mathcal{T}_i)$ then $\Delta(M_j;\, \mathcal{T}_j)$ does not have an M_0-minor. Since the lemma is false $(M_i,\, \mathcal{T}_i)$ cannot be a leaf node, so it has two children. Let us suppose that they are labeled $(M_1^+,\, \mathcal{T}_1 \cup \{T\})$ and $(M_2^+,\, \mathcal{T}_2 \cup \{T\})$.

Assume that $\Delta(M_j^+;\, \mathcal{T}_j)$ has an M_0-minor, for some $j \in \{1,\, 2\}$. Repeated use of Proposition 2.3 tells us that T is a triangle of $\Delta(M_j^+;\, \mathcal{T}_j)$. Since M_0 has no triangles there must be an element $a \in T$ such that $\Delta(M_j^+;\, \mathcal{T}_j)\backslash a$ has an M_0-minor. But

$$\Delta(M_j^+;\, \mathcal{T}_j)\backslash a = \Delta_T(\Delta(M_j^+;\, \mathcal{T}_j))/a = \Delta(M_j^+;\, \mathcal{T}_j \cup \{T\})/a.$$

Therefore $\Delta(M_j^+;\, \mathcal{T}_j \cup \{T\})$ has an M_0-minor, contrary to hypothesis. Thus we conclude that neither $\Delta(M_1^+;\, \mathcal{T}_1)$ nor $\Delta(M_2^+;\, \mathcal{T}_2)$ has an M_0-minor.

Let $M^+ = M_1^+ \oplus_3 M_2^+$. Note that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}_i$. But

$$\Delta(M_1^+;\, \mathcal{T}_1) \oplus_3 \Delta(M_2^+;\, \mathcal{T}_2) = \Delta(M^+;\, \mathcal{T}_i) \tag{4}$$

by Proposition 2.16. Let $P = E(M^+) - E(M_i)$, so that every element in P is parallel to an element in $E(M_i)$, and $M^+\backslash P = M_i$. The method we use to construct the decomposition tree means that no triangle in $\mathcal{T}_i$ contains an element of P. Repeated application of Proposition 2.2 implies that

$$\Delta(M^+;\, \mathcal{T}_i)\backslash P = \Delta(M^+\backslash P;\, \mathcal{T}_i) = \Delta(M_i;\, \mathcal{T}_i).$$

Therefore $\Delta(M^+;\, \mathcal{T}_i)$ has an M_0-minor.

Since neither $\Delta(M_1^+;\, \mathcal{T}_1)$ nor $\Delta(M_2^+;\, \mathcal{T}_2)$ has an M_0-minor, by considering Equation (4) and applying Lemma 2.19 we see that $\Delta_T(\Delta(M_j^+;\, \mathcal{T}_j))$ has an M_0-minor for some $j \in \{1,\, 2\}$. But

$$\Delta_T(\Delta(M_j^+;\, \mathcal{T}_j)) = \Delta(M_j^+;\, \mathcal{T}_j \cup \{T\}),$$

so we have a contradiction to our choice of $(M_i,\, \mathcal{T}_i)$. This completes the proof of the lemma. □

Lemma 4.11. *Suppose that M_0 is a 3-connected binary matroid such that $|E(M_0)| \geq 4$ and M_0 has no triangles. Suppose also that M is a binary matroid such that $\mathrm{si}(M)$ is 3-connected, and that $\mathcal{T}$ is a set of pairwise disjoint triangles of M. Let $\Phi(M,\, \mathcal{T})$ be a decomposition tree of M. If there is a leaf of $\Phi(M,\, \mathcal{T})$ labeled $(M_i,\, \mathcal{T}_i)$ where $\Delta(M_i;\, \mathcal{T}_i)$ has an M_0-minor, then $\Delta(M;\, \mathcal{T})$ has an M_0-minor.*

Proof. The proof is by induction of the size of the decomposition tree. If $\Phi(M,\, \mathcal{T})$ has one node then the result is obvious. Therefore assume that $(M_1^+,\, \mathcal{T}_1 \cup \{T\})$ and $(M_2^+,\, \mathcal{T}_2 \cup \{T\})$ are the children of $(M,\, \mathcal{T})$. We can assume that $(M_i,\, \mathcal{T}_i)$ is a leaf in the decomposition tree $\Phi(M_1^+,\, \mathcal{T}_1 \cup \{T\})$. The inductive hypothesis tells us that $\Delta(M_1^+;\, \mathcal{T}_1 \cup \{T\})$ has an M_0-minor. Let $M^+ = M_1^+ \oplus_3 M_2^+$. Proposition 4.8 says that

$$\Delta(M_1^+;\, \mathcal{T}_1) \oplus_3 \Delta(M_2^+;\, \mathcal{T}_2) = \Delta(M^+;\, \mathcal{T}). \tag{5}$$

Suppose that $T_1, \ldots, T_p$ are the triangles in $\mathcal{T}_2$. We know from Proposition 2.14 that $\mathrm{si}(M_2^+)$ is 3-connected. Let $N^{(0)} = M_2^+$, and for $i = 1, \ldots, p$ let $N^{(i)}$ be $\Delta_{T_i}(N^{(i-1)})$. Thus $N^{(p)} = \Delta(M_2^+;\, \mathcal{T}_2)$. If $\mathrm{si}(\Delta(M_2^+;\, \mathcal{T}_2))$ is not 3-connected then there is an integer $i \in \{1, \ldots, p\}$ such that $\mathrm{si}(N^{(i-1)})$ is 3-connected but $\mathrm{si}(N^{(i)})$ is not. Then there must be a k-separation $(X_1,\, X_2)$ of $N^{(i)}$ where $k < 3$ and $r_{N^{(i)}}(X_j) \geq k$ for $j = 1,\, 2$. We will assume without loss of generality that X_1 contains at least two elements of T_i. Then $(X_1 \cup T_i,\, X_2 - T_i)$ is a k'-separation of $N^{(i)}$ with the property that $k' \leq k$, and $r_{N^{(i)}}(X_1 \cup T_i),\, r_{N^{(i)}}(X_2 - T_i) \geq k'$. Hence we can assume that T_i is contained in X_1. Now $r(N^{(i-1)}) = r(N^{(i)}) - 1$ by Proposition 2.10. It follows from Proposition 2.2 that $\Delta_T(N^{(i-1)}|X_1) = N^{(i)}|X_1$. Therefore $r_{N^{(i-1)}}(X_1) = r_{N^{(i)}}(X_1) - 1$ by Proposition 2.10. Now we see that $(X_1,\, X_2)$ is a k-separation of $N^{(i-1)}$ for some $k < 3$. Moreover $r_{N^{(i-1)}}(X_2) \geq k$, and $r_{N^{(i-1)}}(X_1) \geq 2$ as X_1 contains a triangle of $N^{(i-1)}$. This contradicts the fact that $\mathrm{si}(N^{(i-1)})$ is 3-connected. We conclude that $\mathrm{si}(\Delta(M_2^+;\, \mathcal{T}_2))$ is 3-connected.

Since M_2^+ has at least four rank-one flats (by construction), it follows easily that $\Delta(M_2^+;\, \mathcal{T}_2)$ has at least four rank-one flats. Moreover $\Delta(M_2^+;\, \mathcal{T}_2)$ contains a triangle, so $r(\mathrm{si}(\Delta(M_2^+;\, \mathcal{T}_2))) \geq 3$. No triangle can contain a cocircuit in a 3-connected matroid. Therefore T is a coindependent triangle in $\mathrm{si}(\Delta(M_2^+;\, \mathcal{T}_2))$, so the corank of $\mathrm{si}(\Delta(M_2^+;\, \mathcal{T}_2))$ is at least three. It follows without difficulty from Lemma 2.1 that $\Delta(M_2^+;\, \mathcal{T}_2)$ has a minor M' such that $M' \cong M(K_4)$ and T is a triangle of M'.

By Proposition 2.2 we see that $\Delta(M_1^+;\, \mathcal{T}_1) \oplus_3 \Delta(M_2^+;\, \mathcal{T}_2)$ has $(\Delta(M_1^+;\, \mathcal{T}_1))\triangle(M')$ as a minor. But this last matroid is precisely $\Delta(M_1^+;\, \mathcal{T}_1 \cup \{T\})$, which we know to have an M_0-minor. By considering Equation (5) we conclude that $\Delta(M^+;\, \mathcal{T})$ has an M_0-minor. Since $\mathrm{si}(M^+) \cong M$ repeated application of Proposition 4.9 tells us that $\Delta(M;\, \mathcal{T})$ has an M_0-minor, as desired. □

Corollary 4.12. *Suppose that M_0 is a 3-connected binary matroid such that $|E(M_0)| \geq 4$ and M_0 has no triangles. Suppose also that M is a binary matroid such that $\mathrm{si}(M)$ is 3-connected. Let $\Phi(M,\, \varnothing)$ be a decomposition tree of M and suppose that there is a leaf of $\Phi(M,\, \varnothing)$ labeled $(M_i,\, \mathcal{T}_i)$ where $\Delta(M_i;\, \mathcal{T}_i)$ has an M_0-minor. Then M has an M_0-minor.*

The following lemma (which follows easily from the previous results in this section), should make clear our strategy for deciding whether a binary matroid contains a minor from a particular class.

Lemma 4.13. *Suppose that $\mathcal{M}$ is a collection of internally 4-connected binary matroids such that each matroid in $\mathcal{M}$ has at least four elements, and no matroid in $\mathcal{M}$ contains both a triangle and a triad. Let $\mathcal{M}_\Upsilon$ be the set of matroids in $\mathcal{M}$ that contain at least one triad. Suppose that M is a 3-connected binary matroid and that there is a decomposition tree of M, the leaves of which are labeled $(M_1, \mathcal{T}_1), \ldots, (M_p, \mathcal{T}_p)$. Then M has a minor in $\mathcal{M}$ if and only if:*

(i) there is a leaf $(M_i, \mathcal{T}_i)$ such that M_i has a minor in $\mathcal{M}$; or,
(ii) there is a leaf $(M_i, \mathcal{T}_i)$ such that $\Delta(M_i; \mathcal{T}_i)$ has a minor in $\mathcal{M}_\Upsilon$.

(Indeed Lemma 4.13 is true even if we replace $\mathcal{M}$ in statement (i) of the lemma with $\mathcal{M} - \mathcal{M}_\Upsilon$.) In our case $\mathcal{M}$ is a subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ and either $M(K_{3,3})$ or $M^*(K_{3,3})$ is contained in $\mathcal{M}$. There are two tasks left to consider: For each leaf $(M_i, \mathcal{T}_i)$ of a decomposition tree we must decide whether $\mathrm{si}(M_i)$ has a minor in $\mathcal{M}$, and if this is not the case for any leaf we must decide whether $\Delta(M_i; \mathcal{T}_i)$ has a minor in $\mathcal{M}_\Upsilon$.

The next result was first proved by Tutte [22] (see also [1, Section 7.2]).

Proposition 4.14. *There is a polynomial-time algorithm which, given a binary matroid M, will either return a graph G such that $M(G) = M$, or decide that no such graph exists.*

It follows immediately that we can also decide in polynomial time whether a binary matroid is cographic or planar graphic.

Proposition 4.15. *There is a polynomial-time algorithm which, given a binary matroid M, will decide whether M is isomorphic to Δ_r for some integer $r \geq 3$, or to Υ_r for some even integer $r \geq 4$.*

Proof. We claim that it is possible to decide in polynomial time whether a graph is isomorphic to a Möbius ladder. To see this, we observe that the rim edges of a Möbius ladder are precisely those edges in a unique cycle of length four. Thus we can identify the rim edges in polynomial time and verify that they form a Hamiltonian cycle $v_0, \ldots, v_t$. Let $n = \lfloor t/2 \rfloor$. If t is even and each remaining edge joins a vertex v_i to v_{i+n}, then the graph is a cubic Möbius ladder. If t is odd and each remaining edge joins a vertex v_i to either v_{i+n} or v_{i+n+1}, then the graph is a quartic Möbius ladder. This completes the proof of the claim.

If a binary matroid M is isomorphic to Δ_r then there is an element $e \in E(M)$ such that $M \backslash e$ is isomorphic to $M^*(CM_{2r-2})$, the bond matroid of the cubic Möbius ladder CM_{2r-2}. Thus to decide whether $M \cong \Delta_r$, where $r = r(M)$, we consider each single-element deletion of M in turn and decide in polynomial time whether it is isomorphic to $M^*(CM_{2r-2})$, using the algorithm of Proposition 4.14 and the claim in the previous paragraph.

(Note that if G is a graph such that $M^*(G) = M^*(CM_{2r-2})$ then $G = CM_{2r-2}$ by Whitney's 2-isomorphism theorem.) If $M\backslash e \cong M^*(CM_{2r-2})$ then $M \cong \Delta_r$ if and only if e is in a circuit with all the spoke elements of the Möbius ladder.

If $M \cong \Upsilon_r$ then there is an element $e \in (M)$ such that $M\backslash e \cong M^*(QM_{r-1})$, and e is in a circuit with the spoke elements of the quartic Möbius ladder. Thus a similar argument works in the case of the triadic Möbius matroids. □

Next we consider the problem of identifying when $\Delta(M_i; \mathcal{T}_i)$ has a minor in $\mathcal{M}_\Upsilon$, where $\mathrm{si}(M_i)$ is an internally 4-connected matroid such that $\mathrm{si}(M_i) \in \mathcal{EX}(\mathcal{M})$ and $\mathcal{T}_i$ is a set of pairwise disjoint triangles of M_i.

Proposition 4.16. *Suppose that M is a binary matroid such that $r(M) \geq 3$ and $\mathrm{si}(M)$ is 3-connected. Let $\mathcal{T}$ be a set of pairwise disjoint triangles of M. If there are triangles $T_1, T_2 \in \mathcal{T}$ such that $r_M(T_1 \cup T_2) = 2$ then $\Delta(M; \mathcal{T})$ has an $M(K_{3,3})$-minor.*

Proof. Recall that we take the ground set of $\mathrm{si}(M)$ to be the set of rank-one flats of M. Let $T = \mathrm{cl}_M(T_1)$, so that T is a triangle of $\mathrm{si}(M)$. By the hypotheses we know that $r(\mathrm{si}(M)) \geq 3$. Furthermore $\mathrm{si}(M)$ is 3-connected, so T must be coindependent in $\mathrm{si}(M)$. Therefore $r^*(\mathrm{si}(M)) \geq 3$. Now Lemma 2.1 says that $\mathrm{si}(M)$ has a minor isomorphic to $M(K_4)$ in which T is a triangle. Therefore M has a minor M' isomorphic to the matroid produced from $M(K_4)$ by adding parallel elements to the points in a triangle; and moreover T_1 and T_2 are triangles of M'.

It follows from Proposition 2.2 that $\Delta_{T_2}(\Delta_{T_1}(M'))$ is a minor of $\Delta_{T_2}(\Delta_{T_1}(M))$. However $\Delta_{T_2}(\Delta_{T_1}(M'))$ is isomorphic to $M(K_{3,3})$. Suppose that the members of $\mathcal{T}$ are $T_1, \ldots, T_p$. Let $M^{(0)} = M$, and for $i \in \{1, \ldots, p\}$, let $M^{(i)}$ be $\Delta_{T_i}(M^{(i-1)})$. Thus $M^{(2)}$ has an $M(K_{3,3})$-minor. If $\Delta(M; \mathcal{T})$ does not have an $M(K_{3,3})$-minor then there is an integer $i \in \{3, \ldots, p\}$ such that $M^{(i)}$ has no $M(K_{3,3})$-minor, but $M^{(i-1)}$ does. Since T_i is a triangle of $M^{(i-1)}$ and $M(K_{3,3})$ has no triangles there is an element $a \in T_i$ such that $M^{(i-1)}\backslash a$ has an $M(K_{3,3})$-minor. Proposition 2.9 implies that $M^{(i)}$ has an $M(K_{3,3})$-minor. This contradiction completes the proof. □

Proposition 4.17. *Suppose that M is a cographic matroid such that $r(M) \geq 3$ and $\mathrm{si}(M)$ is internally 4-connected. Let $\mathcal{T}$ be a set of pairwise disjoint triangles of M and suppose that $r_M(T_1 \cup T_2) > 2$ for every pair of distinct triangles $T_1, T_2 \in \mathcal{T}$. Then $\Delta(M; \mathcal{T})$ is cographic, and if M is planar graphic then so is $\Delta(M; \mathcal{T})$.*

Proof. Let G be a graph such that $M = M^*(G)$. We can assume that G contains no isolated vertices, and that furthermore, if e is a loop of M, then there is a connected component of G that contains only the single edge, e. Since M does not consist solely of loops there is a connected component G_0 of G such that G_0 contains more than one edge. Our assumption means that the minimum degree of G_0 is at least two. We shall say that a vertex of degree at least three in G_0 is a *branch vertex* and that a path between two distinct branch vertices is a *branch*. Thus a branch (along with all single-edge components of G) is a rank-one flat of M.

Suppose that T is a triangle of M. Then T is a minimal edge cut-set of G. Let $T = \{e_1, e_2, e_3\}$. Note that no branch of G can contain more than one element of T. For $i = 1, 2, 3$ let l_i be the branch of G that contains e_i, and let $L = \{l_1, l_2, l_3\}$. Then $G_0\backslash L$ contains exactly two connected components. Let A and B be the edge sets of these two components.

Assume that both A and B are non-empty. We initially suppose that both A and B contain at least two branches. If A contains more than two branches then $(A \cup l_1, B \cup \{l_2, l_3\})$ is a 3-separation of M that contradicts the fact that $\mathrm{si}(M)$ is internally 4-connected. Therefore A (and by symmetry, B) contains exactly two branches. Thus $\mathrm{si}(M)$ contains exactly seven elements. As $\mathrm{si}(M)$ is cographic this means that $r(\mathrm{si}(M)) > 3$. Now $\mathrm{si}(M)$ is internally 4-connected, and therefore contains no series pairs or coloops. Therefore $(\mathrm{si}(M))^*$ is a simple graphic matroid with seven elements and rank at most three, a contradiction.

This means that without loss of generality we can assume that A contains precisely one branch. But if v is the end vertex of l_1 that is contained in the subgraph induced by A, then v must be incident with at least two branches in A, so we have a contradiction. Therefore we can assume that A is empty. As G_0 contains no vertices of degree one it follows that there is a degree-three vertex v_T that is incident with l_1, l_2, and l_3.

Suppose that the edge f is contained in l_1 but is distinct from e_1. If we swap the labels on e_1 and f then the bond matroid of the resulting graph is M. If T_1 and T_2 are distinct triangles in $\mathcal{T}$ then $v_{T_1} \neq v_{T_2}$ as $r_M(T_1, T_2) > 2$. Therefore we can assume that every triangle in $\mathcal{T}$ consists of three edges incident with a degree-three vertex. It is clear that if we obtain G' from G by replacing each member of $\mathcal{T}$ with a triangle then $M^*(G') \cong \Delta(M; \mathcal{T})$. Moreover if G is a planar graph then so is G'. □

Recall from Section 2.5 that the ground set of Δ_r, the rank-r triangular Möbius matroid, is $\{e_1, \ldots, e_r, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{r-1}\}$. For $i \in \{1, \ldots, r-1\}$ the elements e_i and a_i are rim elements, while $b_1, \ldots, b_{r-1}$ are spoke elements and e_r is the tip. The only triangles of Δ_r are sets of the form $\{a_i, e_i, e_r\}$ for $1 \leq i \leq r-1$, the sets $\{a_i, a_{i+1}, b_i\}$ and $\{e_i, e_{i+1}, b_i\}$ for $1 \leq i \leq r-2$, and the sets $\{a_1, e_{r-1}, b_{r-1}\}$ and $\{a_{r-1}, e_1, b_{r-1}\}$.

Lemma 4.18. *Suppose that M is a binary matroid such that $\mathrm{si}(M) = \Delta_r$ for some $r \geq 3$. Let $\mathcal{T}$ be a collection of pairwise disjoint triangles of M such that $r_M(T_1 \cup T_2) > 2$ for all pairs of distinct triangles $T_1, T_2 \in \mathcal{T}$. Then $\Delta(M; \mathcal{T})$ has an $M(K_{3,3})$-minor if and only if:*

(i) There is a triangle $T \in \mathcal{T}$ such that $\mathrm{cl}_M(T) = \{a_i, e_i, e_r\}$ for some $1 \leq i \leq r-1$;

(ii) *There are triangles* $T_1, T_2 \in \mathcal{T}$ *such that* $\mathrm{cl}_M(T_1) = \{a_i, a_{i+1}, b_i\}$ *and* $\mathrm{cl}_M(T_2) = \{e_i, e_{i+1}, b_i\}$ *for some* $1 \leq i \leq r-2$*; or,*

(iii) *There are triangles* $T_1, T_2 \in \mathcal{T}$ *such that* $\mathrm{cl}_M(T_1) = \{a_1, e_{r-1}, b_{r-1}\}$ *and* $\mathrm{cl}_M(T_2) = \{a_{r-1}, e_1, b_{r-1}\}$.

Proof. Remember that the ground set of $\mathrm{si}(M)$ is the set of rank-one flats of M. We can assume that M has no loops. Suppose that there is a triangle $T \in \mathcal{T}$ such that $\mathrm{cl}_M(T) = \{a_i, e_i, e_r\}$. Let P be a set of elements such that $M\backslash P \cong \mathrm{si}(M)$, so that $M\backslash P = \Delta_r$. (We are abusing notation here: For example, e_r is both an element of the ground set of $M\backslash P$, and a rank-one flat of M.) We can assume that T is a triangle of $M\backslash P$. Now Claim 4.6 of [12] implies that $\Delta_T(M\backslash P)$ has an $M(K_{3,3})$-minor. Therefore $\Delta_T(M)\backslash P$ has an $M(K_{3,3})$-minor, and so does $\Delta_T(M)$. Suppose that the members of $\mathcal{T}$ are $T_1, \ldots, T_p$. Proposition 2.8 implies that we can assume $T = T_1$. Let $M^{(0)} = M$, and for $i \in \{1, \ldots, p\}$ let $M^{(i)} = \Delta_{T_i}(M^{(i-1)})$. If $\Delta(M; \mathcal{T})$ does not have an $M(K_{3,3})$-minor then there must be an integer i such that $M^{(i)}$ does not have an $M(K_{3,3})$-minor, but $M^{(i-1)}$ does. Now we can obtain a contradiction exactly as in the proof of Proposition 4.16.

Next we will assume that T_1 and T_2 are members of $\mathcal{T}$ such that $\mathrm{cl}_M(T_1) = \{a_i, a_{i+1}, b_i\}$ and $\mathrm{cl}_M(T_2) = \{e_i, e_{i+1}, b_i\}$. Let P be a set of elements such that $M\backslash P = \Delta_r$. We can assume that T_1 is a triangle of $M\backslash P$. Furthermore there is an element $b \in P$ such that $\mathrm{cl}_M(\{b\}) = b_i$. We can assume that $T_2 = \{e_i, e_{i+1}, b\}$ and that T_2 is a triangle of $M\backslash(P-b)$. Claim 4.7 of [12] tells us that $\Delta_{T_2}(\Delta_{T_1}(M\backslash(P-b)))$ has an $M(K_{3,3})$-minor, so $\Delta_{T_2}(\Delta_{T_1}(M))$ has an $M(K_{3,3})$-minor. Exactly as before we can show that $\Delta(M; \mathcal{T})$ has an $M(K_{3,3})$-minor. The same argument shows that $\Delta(M; \mathcal{T})$ has an $M(K_{3,3})$-minor if there are triangles $T_1, T_2 \in \mathcal{T}$ such that $\mathrm{cl}_M(T_1) = \{a_1, e_{r-1}, b_{r-1}\}$ and $\mathrm{cl}_M(T_2) = \{a_{r-1}, e_1, b_{r-1}\}$.

This completes the "if" direction of the proof. To prove the "only if" direction we need to introduce a family of matroids derived from the triangular Möbius matroids. For any positive integer r let Δ_r° be obtained from Δ_r by adding a_i' in parallel to a_i and e_i' in parallel to e_i, for $1 \leq i \leq r-1$.

Claim 4.19. Suppose that N is a restriction of Δ_r° for some $r \geq 3$ and that $\mathcal{T}$ is a set of pairwise disjoint triangles of M with the property that $r_M(T_1 \cup T_2) > 2$ for every pair of distinct triangles $T_1, T_2 \in \mathcal{T}$, and $\mathcal{T}$ does not satisfy conditions (i), (ii), or (iii) of Lemma 4.18. Then $\Delta(N; \mathcal{T})$ is a restriction of Δ_s° for some $s \geq 3$.

Proof. Let $T_1, \ldots, T_p$ be the members of $\mathcal{T}$. The proof is by induction on p. If $\mathcal{T}$ is empty then the result is trivial, so we will assume that $p \geq 1$. Note that T_1 contains b_i for some $i \in \{1, \ldots, r-1\}$, and that no other member of $\mathcal{T}$ contains b_i. Lemma 4.8 of [12] says that $\Delta_{T_1}(\Delta_r^\circ)$ is isomorphic to a restriction of Δ_{r+1}°, and this isomorphism takes spoke elements other than b_i to spoke elements. Since N is a restriction of Δ_r° it follows from Proposition 2.2 that $\Delta_{T_1}(N)$ is isomorphic to a restriction of Δ_{r+1}°, where the isomorphism again takes spoke elements other than b_i to spoke elements. Now it is easy to see that if we apply this isomorphism to $\{T_2, \ldots, T_p\}$ we obtain a collection of pairwise disjoint triangles of Δ°_{r+1} that does not satisfy condition (i), (ii), or (iii) of the lemma. The claim now follows by induction. □

We return to the proof of Lemma 4.18. Note that Δ°_r can be obtained from M by possibly adding and deleting parallel elements. Moreover, if $\mathcal{T}$ is a collection of pairwise disjoint triangles of M such that $\mathcal{T}$ does not satisfy condition (i), (ii), or (iii) of the lemma, then $\mathcal{T}$ is also a collection of pairwise disjoint triangles of Δ°_r. Since Δ_r, and hence Δ°_r does not have an $M(K_{3,3})$-minor for any $r \geq 3$ it follows from Claim 4.19 that $\Delta(\Delta^{\circ}_r; \mathcal{T})$ does not have an $M(K_{3,3})$-minor. Propositions 2.2 and 4.9 imply that $\Delta(M; \mathcal{T})$ has no $M(K_{3,3})$-minor, and this completes the proof. □

Now we prove our main result.

Theorem 1.2. *Suppose that $\mathcal{M}$ is a subset of the family $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ such that $\mathcal{M}$ contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. There is an algorithm which, given a matrix A over* GF(2) *with n columns (and at most n rows), will decide whether $M[A]$ has a minor in $\mathcal{M}$, in time that is bounded by a polynomial function of n.*

Proof. Clearly if we can decide membership in a class of matroids in polynomial time, then we can also decide membership in the dual class. Therefore we will always assume that $M(K_{3,3}) \in \mathcal{M}$. It is not difficult to see that it will suffice to construct a polynomial-time algorithm which will decide membership in $\mathcal{EX}(\mathcal{M})$ when $M[A]$ is 3-connected.

Let $M = M[A]$. By Proposition 4.5 we can construct a decomposition tree $\Phi(M, \varnothing)$ in polynomial-time. First let us assume that $M^*(K_5) \notin \mathcal{M}$. Note that if $(M_i, \mathcal{T}_i)$ is a leaf of $\Phi(M, \varnothing)$ then $|E(\mathrm{si}(M_i))| \leq n$. By Propositions 4.14 and 4.15 we can decide in polynomial time whether $\mathrm{si}(M_i)$ is cographic, planar graphic, or isomorphic to a Möbius matroid, for each leaf $(M_i, \mathcal{T}_i)$ of $\Phi(M, \varnothing)$. Certainly we can decide whether $\mathrm{si}(M_i)$ is isomorphic to one of a finite number of fixed matroids. Therefore, by examining Theorems 1.1, 3.2, 3.5, and 3.7, we see that we can decide in polynomial time whether M_i has a minor in $\mathcal{M}$, for each leaf $(M_i, \mathcal{T}_i)$. If some M_i has a minor in $\mathcal{M}$, then so does M, by Proposition 4.6, in which case we can stop. Therefore we suppose that each M_i belongs to $\mathcal{EX}(\mathcal{M})$.

Since the only matroid in $\mathcal{M}$ with any triads is $M(K_{3,3})$, it follows from Proposition 4.7 that if M has a minor in $\mathcal{M}$ then M must have an $M(K_{3,3})$-minor. By Lemma 4.10 and Corollary 4.12 this is true if and only if there is a leaf $(M_i, \mathcal{T}_i)$ such that $\Delta(M_i; \mathcal{T}_i)$ has an $M(K_{3,3})$-minor. Note that by definition of 3-sum it follows that $r(\mathrm{si}(M)) \geq 3$ for every leaf $(M_i, \mathcal{T}_i)$. Propositions 4.16 and 4.17 imply that if $\mathrm{si}(M)$ is cographic or planar graphic then $\Delta(M_i; \mathcal{T}_i)$ has an $M(K_{3,3})$-minor if and only there is a pair of distinct triangles in $\mathcal{T}_i$ whose union has rank two. Certainly we can decide whether this is true in polynomial time. Note that the triadic Möbius matroids have no triangles. If $\mathrm{si}(M_i)$ is a triangular Möbius matroid then we can decide whether $\Delta(M_i;\, \mathcal{T}_i)$ has an $M(K_{3,3})$-minor by examining $\mathcal{T}_i$ and using Proposition 4.16 and Lemma 4.18. Suppose that $\mathrm{si}(M)$ is isomorphic to one of a finite number of fixed matroids. If there are two distinct triangles in $\mathcal{T}_i$ whose union has rank two then $\Delta(M_i;\, \mathcal{T}_i)$ has an $M(K_{3,3})$-minor. Therefore we will assume that this is not the case. It follows that $|\mathcal{T}_i|$ is bounded by some fixed constant. Now we can obtain the matroid M' by adding parallel elements to $\mathrm{si}(M_i)$ in such a way that there is a set $\mathcal{T}$ of pairwise disjoint triangles in M', such that there is a natural bijection between triangles in $\mathcal{T}$ and triangles in $\mathcal{T}_i$. Moreover $|E(M')|$ is bounded by some constant. Therefore we can decide in constant time whether $\Delta(M';\, \mathcal{T})$ has an $M(K_{3,3})$-minor, and since M_i is obtained from M' by possibly adding parallel elements it follows from Propositions 2.2 and 4.9 that $\Delta(M';\, \mathcal{T})$ has an $M(K_{3,3})$-minor if and only if $\Delta(M_i;\, \mathcal{T}_i)$ does.

This completes the proof of the theorem in the case that $\mathcal{M}$ does not contain $M^*(K_5)$. If $M^*(K_5) \in \mathcal{M}$ then we proceed as before and construct a decomposition tree $\Phi(M,\, \varnothing)$. By examining Theorem 3.4, Theorem 3.6, and Theorem 3.8, we see that to decide whether $M_i \in \mathcal{EX}(\mathcal{M})$ for each leaf $(M_i,\, \mathcal{T}_i)$ we need only decide whether $\mathrm{si}(M_i)$ is planar graphic or isomorphic to one of a finite set of sporadic matroids. This can clearly be done in polynomial time. Once again we will assume that $M_i \in \mathcal{EX}(\mathcal{M})$ for each leaf $(M_i,\, \mathcal{T}_i)$, for otherwise we are done. Now Lemma 4.10 and Corollary 4.12 imply that M has a minor in $\mathcal{M}$ if and only if there is some leaf $(M_i,\, \mathcal{T}_i)$ such that $\Delta(M_i;\, \mathcal{T}_i)$ has either an $M(K_{3,3})$-minor, or an $M^*(K_5)$-minor. We can decide whether this is true in constant time if $\mathrm{si}(M_i)$ is isomorphic to one of a set of sporadic matroids. Therefore the only thing left to do is decide whether $\Delta(M_i;\, \mathcal{T}_i)$ has a minor isomorphic to $M(K_{3,3})$ or $M^*(K_5)$ in the case that $\mathrm{si}(M_i)$ (and hence M_i) is planar graphic.

If there are distinct triangles $T_1,\, T_2 \in \mathcal{T}_i$ such that $r_{M_i}(T_1 \cup T_2) = 2$ then $\Delta(M_i;\, \mathcal{T}_i)$ has an $M(K_{3,3})$-minor by Proposition 4.16. If no two triangles in $\mathcal{T}$ satisfy this condition then $\Delta(M_i;\, \mathcal{T}_i)$ is planar graphic by Proposition 4.17, and hence has no minor isomorphic to $M(K_{3,3})$ or $M^*(K_5)$. This completes the proof of the theorem. □

We conclude this section by noting that Möbius matroids have branch-width four, so by using Hliněný's [7] matroid analogue of Courcelle's [3] theorem, the minor-testing algorithm for graphs, due to Robertson and Seymour [17], and Theorem 1.1, we can decide in polynomial-time whether a represented internally 4-connected binary matroid belongs to any class of the form $\mathcal{EX}(\mathcal{N} \cup \{M(K_{3,3})\})$, where $\mathcal{N}$ is any set of binary matroids. However, any such algorithm relies upon (and in fact implies) a non-constructive result from the Graph Minors project of Robertson and Seymour. Therefore, this fact does not make Theorem 1.2 redundant, as that result is constructive, in the sense that the algorithms it describes could be implemented without too much difficulty.

4.2. **Oracle algorithms.** Up to this point we have assumed that all binary matroids are represented by matrices over GF(2). In this section we will assume that a matroid M (not necessarily binary) is represented by an oracle. Suppose that $\mathcal{M}$ is a subcollection of the family $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ and that $\mathcal{M}$ contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. If we wish to decide whether M has a minor in $\mathcal{M}$, then we can construct a partial representation A of M, and run the algorithm of Theorem 1.2 upon the binary matroid $M[I_{r(M)}|A]$. If M is a binary matroid then this algorithm will return the correct answer. However if M is non-binary then we cannot be certain that this is the case. If the algorithm indicates that $M[I_{r(M)}|A]$ has a minor in $\mathcal{M}$ then all we can be sure of is that either M has a minor in $\mathcal{M}$, or M is non-binary. Similarly, if the algorithm decides that $M[I_{r(M)}|A]$ has no minor in $\mathcal{M}$ then either M has no minor in $\mathcal{M}$, or M is non-binary. Thus we can decide whether M has a minor in $\mathcal{M} \cup \{U_{2,4}\}$, but we cannot decide which of these matroids it has as a minor.

In some sense this is the best we can hope for, as a famous example of Seymour's [19] shows: For $r \geq 3$ let $\{e_1, \ldots, e_r\}$ be the standard basis of the vector space over GF(2) with dimension r. Let d be the sum of $e_1, \ldots, e_r$, and for $1 \leq i \leq r$ let d_i be the sum of d and e_i. Let N_r be the binary matroid represented by the set $\{e_1, \ldots, e_r, d_1, \ldots, d_r\}$. If H is a subset of $E(N_r)$ such that $|H \cap \{d_1, \ldots, d_r\}|$ is odd and $|H \cap \{e_i, d_i\}| = 1$ for $1 \leq i \leq r$, then H is a circuit-hyperplane of N_r. Let $N_r(H)$ be the matroid obtained from N_r by relaxing H. It is not difficult to prove by induction on r that N_r has no minor in the set $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$. Moreover, $N_r(H)$ is non-binary. In the worst case, an oracle algorithm will have to check each of the 2^{r-1} candidate sets H to decide whether the matroid it is considering is isomorphic to N_r, or $N_r(H)$. Therefore we have the following result.

Proposition 4.20. *Let $\mathcal{M}$ be a subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$. There is no polynomial function p such that one can decide whether a matroid M belongs to $\mathcal{EX}(\mathcal{M})$ using at most $p(|E(M)|)$ calls to an oracle.*

We note that the binary matroid N_r contains many 3-separations, and is therefore far from being internally 4-connected. If we restrict our attention to internally 4-connected matroids the situation changes dramatically. Seymour [19] shows that there is a polynomial p and an algorithm which, given a matroid M (not necessarily binary), will either output a graph G such that $M = M(G)$, or decide that no such graph exists, using at most $p(|E(M)|)$ calls to an oracle. Using a similar strategy to that in the proof of Proposition 4.15 we can show that it is possible to decide whether a matroid M is isomorphic to a Möbius matroid, using only a polynomial number of calls to an oracle. Since it is obviously possible to decide whether a matroid

M is isomorphic to one of a finite number of sporadic matroids using a constant number of oracle calls, the next proposition follows from the results of Section 3.

Proposition 4.21. *Let $\mathcal{M}$ be a subset of $\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$ that contains either $M(K_{3,3})$ or $M^*(K_{3,3})$. There is a polynomial function p such that one can decide whether an internally 4-connected matroid M belongs to $\mathcal{EX}(\mathcal{M})$, using at most $p(|E(M)|)$ calls to an oracle.*

5. Maximum-sized binary matroids with no $M(K_{3,3})$

Suppose that $\mathcal{M}$ is a family of matroids. We say that $M \in \mathcal{M}$ is a *maximum-sized member* of $\mathcal{M}$ if M is simple, and whenever $M' \in \mathcal{M}$ is a simple matroid with the same rank as M, then $|E(M')| \leq |E(M)|$.

Kung [10] investigated the maximum-sized members of $\mathcal{EX}(M(K_{3,3}))$, and showed that such a matroid M satisfies $|E(M)| \leq 10r(M)$. We complete this programme by characterizing the maximum-sized members of $\mathcal{EX}(M(K_{3,3}))$. As a consequence, we show that if M is a rank-r maximum-sized member of $\mathcal{EX}(M(K_{3,3}))$, then $|E(M)| = 14r/3 - \alpha(r)$, where $\alpha(r)$ takes on the values 7, 11/3, and 19/3 according to the residue of r modulo 3.

Suppose that $M \in \mathcal{EX}(M(K_{3,3}))$ and that T is a triangle of M. We say that T is an *allowable triangle* of M if $\Delta_T(M)$ has no $M(K_{3,3})$-minor.

Lemma 5.1. *Suppose that M is a 3-connected member of $\mathcal{EX}(M(K_{3,3}))$. If either*

(i) M has an allowable triangle; or,

(ii) $r(M) > 4$,

then $|E(M)| \leq 4r(M) - 5$.

Proof. Let M be a 3-connected member of $\mathcal{EX}(M(K_{3,3}))$. Let $r = r(M)$. Note that if M satisfies statement (i) or (ii), then $r \geq 2$. Moreover, if $r = 2$, then $|E(M)| = 3$, so $|E(M)| \leq 4r - 5$. Henceforth we assume that $r \geq 3$. Suppose that M is internally 4-connected. Theorem 1.1 implies that M is either cographic, isomorphic to a Möbius matroid, or isomorphic to one of the sporadic matroids in Theorem 1.1.

Suppose that M is a sporadic matroid. There are no sporadic matroids with rank 3. If $r > 4$ then it is easily confirmed that $|E(M)| \leq 4r - 5$. If $r = 4$ and M has an allowable triangle, then $|E(M)| \leq 11$ (see Appendix C of [12]), and therefore $|E(M)| \leq 4r - 5$.

Suppose that M is a Möbius matroid. A rank-r triangular Möbius matroid contains $3r - 2$ elements, and a rank-r triadic Möbius matroid has $2r - 1$ elements. Since $r \geq 3$ it follows that $|E(M)| \leq 4r - 5$.

Now assume that M is cographic. Suppose that G is a graph such that $M = M^*(G)$. As $r \geq 3$, it follows that the minimum degree of G is at least three, because M is internally 4-connected. By splitting vertices, we can see that $|E(M)|$ is no greater than the number of elements in a rank-r cographic matroid corresponding to a graph that is regular with degree three. Such a cographic matroid contains exactly $3r-3$ elements. It again follows that $|E(M)| \leq 4r-5$.

Now we assume that the lemma is false, and that M is a counterexample with the smallest possible rank. The previous paragraphs imply that M is not internally 4-connected.

Let (X_1, X_2) be an exact 3-separation of M such that $|X_i| \geq 4$ for $i = 1, 2$. By Proposition 2.12 there are binary matroids M_1 and M_2 on the ground sets $X_1 \cup T$ and $X_2 \cup T$ respectively such that $T \cap (X_1 \cup X_2) = \varnothing$ and $M = M_1 \oplus_3 M_2$. Assume that the ranks of M_1 and M_2 are r_1 and r_2 respectively. Since the 3-sum $M_1 \oplus_3 M_2$ is defined, it follows that T does not contain a cocircuit in either M_1 or M_2. Thus $r_i = r_M(X_i)$ for $i = 1, 2$. Therefore $r_1 + r_2 - r = 2$. Because $|E(M_i)| \geq 7$ and every parallel pair of M_i involves a member of T it follows that $r_i > 2$ for $i = 1, 2$. Hence $r_1, r_2 < r$. Proposition 2.14 says that $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are 3-connected. Moreover M_i (and hence $\mathrm{si}(M_i)$) has no $M(K_{3,3})$-minor for $i = 1, 2$ by Proposition 2.13. Therefore we can apply the lemma to $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ by our inductive assumption.

If T is not an allowable triangle of M_i then $\Delta_T(M_i)$ has an $M(K_{3,3})$-minor, and therefore M has an $M(K_{3,3})$-minor by Proposition 2.15, a contradiction. Therefore T is an allowable triangle in both M_1 and M_2. For $i = 1, 2$ let $n_i = |E(\mathrm{si}(M_i))|$. As the lemma holds for $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ we deduce that $n_i \leq 4r_i - 5$ for $i = 1, 2$.

The only parallel classes of M_i have size two, and contain an element of T. Note that no element in T can be in a parallel pair in both M_1 and M_2, for that would imply that $M = M_1 \oplus_3 M_2$ has a parallel pair. Let m be the number of elements in T that are in a parallel pair in either M_1 or M_2. Then

$$\begin{aligned}|E(M)| &= n_1 + n_2 - m - 2(3-m) = n_1 + n_2 + m - 6 \\ &\leq (4r_1 - 5) + (4r_2 - 5) - 3 = 4(r_1 + r_2 - 2) - 5 = 4r - 5.\end{aligned}$$

Therefore M is not a counterexample to the lemma. This contradiction completes the proof. □

For integers $r \geq 2$ we recursively define the classes of rank-r matroids $\mathcal{P}_r$ as follows:

$$\mathcal{P}_2 = \{\mathrm{PG}(1,\,2)\}, \quad \mathcal{P}_3 = \{\mathrm{PG}(2,\,2)\}, \quad \mathcal{P}_4 = \{\mathrm{PG}(3,\,2)\}$$

and for $r > 4$ we define $\mathcal{P}_r$ to be the set

$$\{P(M,\,\mathrm{PG}(3,2)) \mid M \in \mathcal{P}_{r-3}\},$$

where $P(M,\,\mathrm{PG}(3,2))$ is the parallel connection of M and $\mathrm{PG}(3,2)$ (see [15, Section 7.1]) along an arbitrary basepoint. It is well known that parallel connections can be expressed as 2-sums. It follows that if M_0 is a 3-connected binary matroid, and neither M_1 nor M_2 has M_0 as a minor, the parallel connection of M_1 and M_2 does not have an M_0-minor ([12, Proposition 2.20]). The next result follows from this fact and induction.

Proposition 5.2. *Let $r \geq 2$ be an integer, and suppose that $M \in \mathcal{P}_r$. Then M has no $M(K_{3,3})$-minor.*

Note that for each $r \geq 2$ the matroids in $\mathcal{P}_r$ have the same size. Let $f(r)$ denote this common size. The precise value of $f(r)$ depends on the residue class of r modulo 3, as shown in Table 1.

Rank r	Size	$f(r)$
$r = 3k$	$14k - 7$	$14r/3 - 7$
$r = 3k + 1$	$14k + 1$	$14r/3 - 11/3$
$r = 3k + 2$	$14k + 3$	$14r/3 - 19/3$

TABLE 1. The size of matroids in $\mathcal{P}_r$.

Define $\alpha(r) = 14r/3 - f(r)$ so that each member of $\mathcal{P}_r$ has $14r/3 - \alpha(r)$ elements and $\alpha(r)$ depends only on the value of r modulo 3.

Now we can prove the main result of this section.

Theorem 5.3. *A matroid M with rank $r \geq 2$ is a maximum-sized member of $\mathcal{EX}(M(K_{3,3}))$ if and only if $M \in \mathcal{P}_r$.*

Proof. We prove the "only if" direction first. Assume that M is a maximum-sized member of $\mathcal{EX}(M(K_{3,3}))$ and that M does not belong to $\mathcal{P}_r$ for any $r \geq 2$. Let $r = r(M)$ and assume that no such counterexample exists with rank less than r. The theorem is easily seen to be true for matroids of rank at most four, so $r > 4$. Obviously M is at least as large as a member of $\mathcal{P}_r$, so $|E(M)| \geq 14r/3 - 7$. As $r > 4$ we see that $14r/3 - 7 > 4r - 5$, so M is not 3-connected by Lemma 5.1. Hence there is an exact k-separation (X_1, X_2) of M, where $k < 3$.

If $k = 1$ then $M = M_1 \oplus M_2$, where $M_i = M|X_i$ for $i = 1, 2$. Neither M_1 nor M_2 has an $M(K_{3,3})$-minor, and certainly both are simple. Suppose that $r_i = r(M_i)$ for $i = 1, 2$, so that $r = r_1 + r_2$. Since M is simple, $r_i > 0$ and hence $r_i < r$ for $i = 1, 2$. Therefore we can apply our inductive hypothesis and conclude that $|E(M_i)| \leq 14r_i/3 - \alpha(r_i)$ for $i = 1, 2$. Now

$$|E(M)| = |E(M_1)| + |E(M_2)| \leq 14r/3 - (\alpha(r_1) + \alpha(r_2)).$$

But $\alpha(r_1) + \alpha(r_2) > 7$, regardless of the residue classes of r_1 and r_2 modulo 3, so $|E(M)| < 14r/3 - 7$, contradicting our earlier conclusion.

Now we can assume that M is connected, and that $k = 2$. Then Proposition 2.11 says that $M = M_1 \oplus_2 M_2$, where the ground set of M_i is $X_i \cup p$ for $i = 1, 2$, and $p \notin X_1 \cup X_2$. Let $r_i = r(M_i)$ for $i = 1, 2$. As p is not a loop or coloop in M_1 or M_2 it follows from Proposition 2.3 that $r_i = r_M(X_i)$ for $i = 1, 2$. Thus $r_1 + r_2 - r = 1$. As M has no parallel pairs it follows that $r_1, r_2 \geq 2$. Therefore $r_1, r_2 < r$. Proposition 2.3 also implies that neither M_1 nor M_2 has an $M(K_{3,3})$-minor.

The fact that M has no parallel pairs means that for all $i \in \{1, 2\}$, either M_i is simple or M_i contains no loops and exactly one parallel pair, which contains p. Suppose that both M_1 and M_2 are simple. Let M_1' be obtained by adding an element in parallel to p. Then $M_1' \oplus_2 M_2$ is simple, and has no $M(K_{3,3})$-minor by [12, Proposition 2.20]. Moreover $M_1' \oplus_2 M_2$ has more elements than $M = M_1 \oplus_2 M_2$, a contradiction. Therefore either M_1 or M_2 contains a single parallel pair, and this pair contains p. It cannot be the case that both M_1 and M_2 contain a parallel pair, for then M would have a parallel pair.

Suppose that M_1 is simple, and that M' is a simple rank-r_1 matroid in $\mathcal{EX}(M(K_{3,3}))$ such that $|E(M')| > |E(M_1)|$. Let us assume that $E(M') \cap E(M_2) = \{p\}$. We can also assume that p is not a coloop of M', as M' is not isomorphic to U_{r_1,r_1}. Therefore $M' \oplus_2 M_2$ is defined, and has no $M(K_{3,3})$-minor by [12, Proposition 2.20]. Moreover $M' \oplus_2 M_2$ is simple and $|E(M' \oplus_2 M_2)| > |E(M)|$, a contradiction. Next we suppose that M_1 contains a single parallel pair. Assume that $M' \in \mathcal{EX}(M(K_{3,3}))$ is a simple rank-r_1 matroid and that $|E(M')| > |E(\mathrm{si}(M_1))|$. We can assume that the 2-sum of M' and M_2 along the basepoint p is defined. Let M'' be the matroid obtained from M' by adding p' in parallel to p. From our earlier discussion we see that p is not in a parallel pair in M_2. It follows that $M'' \oplus_2 M_2$ is simple. Moreover $M'' \oplus_2 M_2$ has no $M(K_{3,3})$-minor. Furthermore $M'' \oplus_2 M_2$ contains strictly more elements than M. In either case we have a contradiction, so $\mathrm{si}(M_1)$ is a maximum-sized member of $\mathcal{EX}(M(K_{3,3}))$. By the same argument, so is $\mathrm{si}(M_2)$.

The inductive hypothesis implies that $\mathrm{si}(M_i) \in \mathcal{P}_{r_i}$ for $i = 1, 2$. We have already noted that p is not in a parallel pair in both M_1 and M_2. From this we can deduce that $|E(M)| \leq |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 1$. As M must be at least as large as a member of $\mathcal{P}_r$ we deduce that

$$14r/3 - \alpha(r) \leq |E(\mathrm{si}(M_1))| + |E(\mathrm{si}(M_2))| - 1.$$

However, by our inductive assumption we see that $|E(\mathrm{si}(M_i))| \leq 14r_i/3 - \alpha(r_i)$ for $i = 1, 2$. Therefore

$$\begin{aligned} 14r/3 - \alpha(r) &\leq 14/3(r_1 + r_2) - (\alpha(r_1) + \alpha(r_2)) - 1 \\ &= 14/3(r + 1) - (\alpha(r_1) + \alpha(r_2)) - 1. \end{aligned}$$

This implies that $\alpha(r_1) + \alpha(r_2) - \alpha(r) \leq 11/3$. Combining this with the fact that $r_1 + r_2 = r + 1$, and examining the cases, we see that either r_1 or r_2 must be congruent to 1 modulo 3. Without loss of generality we will assume that $r_1 \equiv 1 \pmod 3$. Since $\mathrm{si}(M_1) \in \mathcal{P}_{r_1}$ this means that $\mathrm{si}(M_1)$ is obtained by taking the parallel connection of a collection of isomorphic copies of PG(3, 2). It follows easily from Proposition 2.4 that the parallel connection is an associative operation, and M is the parallel connection of $\mathrm{si}(M_1)$ and a matroid from $\mathcal{P}_{r_2}$. Therefore M belongs to $\mathcal{P}_r$, a contradiction.

To prove the other direction of the theorem we note that if $M \in \mathcal{P}_r$ for some $r \geq 2$ then M is simple and $M \in \mathcal{EX}(M(K_{3,3}))$ by Proposition 5.2. By the first part of the proof any maximum-sized rank-r member of $\mathcal{EX}(M(K_{3,3}))$ has the same size as M, and therefore M itself is maximum-sized. This completes the proof. □

6. Critical exponents

Suppose that M is a loopless GF(q)-representable matroid with rank r. Then M can be considered as a multiset of points in the projective space PG($r-1$, q). The *critical exponent* of M over q is the smallest integer k such that there is a set of hyperplanes $H_1, \ldots, H_k$ in PG($r-1$, q) with the property that $H_1 \cap \cdots \cap H_k$ contains no points of $E(M)$. Finding $c(M;\, q)$ is known as the *critical problem* (see Kung [11] for an exposition). It is easy to see that if e is an element of the matroid M, then $c(M\backslash e;\, q) \leq c(M;\, q)$, for all possible q.

Kung [10] looked at the critical exponent over GF(2) of binary matroids with no $M(K_{3,3})$-minor. He showed that if $M \in \mathcal{EX}(M(K_{3,3}))$ is simple then $c(M;\, 2) \leq 10$. In this section we show that in fact $c(M;\, 2) \leq 4$, and that this bound can not be improved. In particular, we show that if $M \in \mathcal{EX}(M(K_{3,3}))$ is loopless and $c(M;\, 2) = 4$, then M has a 3-connected component isomorphic to PG(3, 2).

The critical exponent can be expressed in terms of the *characteristic polynomial*. Suppose that M is a matroid on the ground set E. Then the characteristic polynomial of M, denoted $\chi(M;\, t)$, is defined by

$$\chi(M;\, t) = \sum_{A \subseteq E} (-1)^{|A|} t^{r(M)-r(A)}.$$

Suppose that M is a loopless GF(q)-representable matroid. Then $\chi(M;\, q^k) \geq 0$ for every positive integer k, and $c(M;\, q)$ is the least positive integer k such that $\chi(M;\, q^k)$ is positive. It is known that if G is a graph, then the *flow polynomial* of G, denoted $F(G;\, t)$, is equal to $\chi(M^*(G);\, t)$.

Lemma 6.1. *Suppose that M is an internally* 4*-connected member of $\mathcal{EX}(M(K_{3,3}))$. Then $c(M;\, 2) \leq 4$, and if $c(M;\, 2) = 4$, then M is isomorphic to* PG(3, 2)*.*

Proof. It is easy to see that PG(3, 2) has critical exponent 4 over GF(2) (see [11, Section 8.1]).

Assume that M is an internally 4-connected member of $\mathcal{EX}(M(K_{3,3}))$ other than PG(3, 2). Suppose that $M = M^*(G)$ is a cographic matroid. Jaeger's 8-flow theorem [8] shows that $\chi(M;\, 8) = F(G;\, 8) > 0$, and hence $c(M; 2) \leq 3$. Now suppose that $M \cong \Delta_r$, a triangular Möbius matroid. Consider the GF(2)-representation of Δ_r discussed in Section 2.5. Suppose that each point of PG($r-1$, 2) corresponds to a vector $(x_1, \ldots, x_r)$. Let H_1, H_2, and H_3 be hyperplanes of PG($r-1$, 2) defined, respectively, by the equations $x_r = 0$, $x_1 + \cdots + x_r = 0$, and either $x_1 + x_3 + \cdots + x_{r-1} = 0$ or $x_1 + x_3 + \cdots + x_{r-2} = 0$, depending on whether r is even or odd. It is easy to see that no point of Δ_r is contained in all of H_1, H_2, and H_3, so $c(M;\, 2) \leq 3$.

Suppose that $M \cong \Upsilon_r$, a triadic Möbius matroid. Consider the representation of Υ_r in Section 2.5. No point of Υ_r is contained in the hyperplane defined by $x_1 + \cdots + x_r = 0$, so $c(M; 2) \leq 1$.

Finally we suppose that M is isomorphic to one of the sporadic matroids in Theorem 1.1. The largest such matroid with rank 4 is PG(3, 2), and it known that every proper minor of this matroid has critical exponent at most three over GF(2) [11, Section 8.1]. Thus we will suppose that $r(M) \geq 5$. The sporadic matroid T_{12} has rank 6. By examining the matrix representation of T_{12} in [12, Appendix B], we see that no point of T_{12} is contained in the hyperplane defined by $x_1 + \cdots + x_6 = 0$. Thus $c(T_{12}; 2) \leq 1$. Let A be the matrix in [12, Appendix B] such that $[I_5|A]$ represents the rank-5 sporadic matroid $M^a_{5,12}$. If

$$H^a_{5,12} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

then $H^a_{5,12}[I_5|A]$ contains no zero columns. This means that no point of $M^a_{5,12}$ is contained in all three of the hyperplanes defined by $x_1 + x_2 = 0$, $x_1 + x_3 + x_5 = 0$, and $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Thus $c(M^a_{5,12}; 2) \leq 3$.

In the same way we can show that $M_{5,13}$, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, and $M_{11,21}$ all have critical exponent at most three by examining the matrices

$$H_{5,13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad H_{6,13} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$H_{7,15} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad H_{9,18} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$H_{11,21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

As every sporadic matroid in Theorem 1.1 can be produced from one of PG(3, 2), $M^a_{5,12}$, $M_{5,13}$, T_{12}, $M_{6,13}$, $M_{7,15}$, $M_{9,18}$, or $M_{11,21}$ by deleting elements, the proof is complete. □

Next we come to the main result of this section.

Theorem 6.2. *Suppose that $M \in \mathcal{EX}(M(K_{3,3}))$ is a loopless matroid. Then $c(M; 2) \leq 4$, and if $c(M; 2) = 4$, then either*

(i) M is isomorphic to PG(3, 2)*; or,*

(ii) M can be expressed as the 1- *or* 2-*sum of M_1 and M_2, where M_1, M_2 belong to $\mathcal{EX}(M(K_{3,3}))$, and either $c(M_1; 2) = 4$ or $c(M_2; 2) = 4$.*

Proof. Suppose that M is a minor-minimal counterexample to the theorem. Lemma 6.1 shows that M cannot be internally 4-connected. Assume that M is not connected, so that M can be expressed as $M_1 \oplus M_2$. Clearly M_1

and M_2 are loopless members of $\mathcal{EX}(M(K_{3,3}))$. It is well known, and easy to verify, that $\chi(M;\, t) = \chi(M_1;\, t)\chi(M_2;\, t)$. If $c(M;\, 2) > 4$ then $\chi(M;\, 16) = 0$, implying that $\chi(M_i;\, 16) = 0$ for some $i \in \{1,\, 2\}$. But M_i is a proper minor of M, so we have a contradiction to minimality. Therefore we suppose that $c(M;\, 2) = 4$, so that $\chi(M;\, 8) = 0$. This implies that $c(M_i;\, 2) = 4$ for some $i \in \{1,\, 2\}$, and M satisfies statement (ii).

Now we must assume that M is connected. Suppose that M can be expressed as the 2-sum of M_1 and M_2 along the basepoint p, by Proposition 2.11. Again M_1 and M_2 are loopless members of $\mathcal{EX}(M(K_{3,3}))$. Walton and Welsh [24, (7)] note that the following relation holds:

$$\chi(M;\, t) = \frac{\chi(M_1;\, t)\chi(M_2;\, t)}{t-1} + \chi(M_1/p;\, t)\chi(M_2/p;\, t). \tag{6}$$

Since M_1 and M_2 are isomorphic to minors of M it follows that $\chi(M_i;\, 16) > 0$ for $i = 1,\, 2$. Now (6) implies that $\chi(M;\, 16) > 0$, so $c(M;\, 2) \leq 4$. Therefore it must be the case that $c(M;\, 2) = 4$, so that $\chi(M;\, 8) = 0$. Since $\chi(M_1;\, 8) \geq 0$ and $\chi(M_2;\, 8) \geq 0$ it follows that either $\chi(M_1;\, 8) = 0$ or $\chi(M_2;\, 8) = 0$. Then M satisfies statement (ii) of the theorem, so we have a contradiction.

Now we must assume that M is 3-connected, so $M = M_1 \oplus_3 M_2$ for some matroids M_1 and M_2. Proposition 2.13 implies that M_1 and M_2 are isomorphic to proper minors of M. Moreover M_1 and M_2 are loopless, and both $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are 3-connected by Proposition 2.14.

Suppose that the 3-sum of M_1 and M_2 is along the triangle T, where $T = \{a,\, b,\, c\}$. The following equality is from Walton and Welsh [24].

$$\begin{aligned}\chi(M;\, t) = \frac{\chi(M_1;\, t)\chi(M_2;\, t)}{(t-1)(t-2)} &+ \chi(M\backslash a\backslash b/c;\, t)\\ &+ \chi(M\backslash a/b;\, t) + \chi(M/a;\, t).\end{aligned} \tag{7}$$

All the matroids M_1, M_2, $M\backslash a\backslash b/c$, $M\backslash a/b$, and M/a have critical exponent at most four, so the characteristic polynomial of each of these matroids, evaluated at 16, produces a positive answer. Therefore $\chi(M;\, 16) > 0$ and $c(M;\, 2) \leq 4$.

It must be the case that $c(M;\, 2) = 4$, so that $\chi(M;\, 8) = 0$. Then the terms of the sum in (7) must be zero at the point $t = 8$. In particular, we can assume by relabeling that $\chi(M_1;\, 8) = 0$ and $c(M_1;\, 2) = 4$.

The critical exponent of $\mathrm{si}(M_1)$ is precisely the critical exponent of M_1. Since $\mathrm{si}(M_1)$ is 3-connected and obeys the theorem, it follows that $\mathrm{si}(M_1)$ is isomorphic to $\mathrm{PG}(3,\, 2)$. Now $\mathrm{PG}(3,\, 2)$ has no allowable triangles (see [12, Appendix C]). Therefore $\Delta_T(\mathrm{si}(M_1))$ has an $M(K_{3,3})$-minor, so $\Delta_T(M_1)$ has an $M(K_{3,3})$-minor. Proposition 2.15 implies that $\Delta_T(M_1)$ is isomorphic to a minor of M, so M has an $M(K_{3,3})$-minor. This contradiction completes the proof. $\square$

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