Invariants for Multi-Twists, Screw Systems and Serial Manipulators

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Abstract The Euclidean group of proper isometries $SE(3)$ acts on its Lie algebra, the vector space of twists by the adjoint action. This extends to multi-twists and screw systems. Invariants of these actions encode geometric information about the objects and are fundamental in applications to robot kinematics. This paper explores relations between known invariants and applies them to serial manipulators.

Key words: Euclidean group, adjoint action, multi-twists, screw systems, invariant polynomials, serial manipulators.

1 Introduction

Twists, vectors of twists and screw systems all play an important role in mathematical models of robot manipulator kinematics. They describe the infinitesimal capabilities of joints, links, and end-effector or platform for a given configuration of a manipulator. Properties of these objects, especially those that are invariant under the Euclidean group, are therefore fundamental. By way of examples:

1. a 1-degree of freedom (dof) holonomic, lower-pair joint may be represented by a twist or screw, which remains invariant under motions of the joint and simultaneous motion of the set of links (and their joints) in space;
2. the Denavit–Hartenberg parameters for serial manipulators are design parameters that are assumed to be unchanged under arbitrary movements of the manipulator;
3. the infinitesimal motion of the platform or end-effector of a manipulator with $k < 6$-dof is, in a given configuration, described by a k-system, which should be independent of the choice of coordinates.

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The aim of this paper is to bring together information about polynomial invariants in each of these situations. A central role is given to the multi-twist invariants [5, 6] and their connection to the invariants of Selig [12].

2 The Euclidean Group: Twists, Multi-Twists and Screw Systems

The displacements of a spatial rigid body are described by the (special) Euclidean group $SE(3)$, a 6-dimensional Lie group, identified via choice of coordinate frames as the semi-direct product of rotations about the origin, $SO(3)$, and translations $\mathbb{R}^3$. The action of $(A,\mathbf{a}) \in SO(3) \ltimes \mathbb{R}^3 \cong SE(3)$ transforms body coordinates $\mathbf{x} \in \mathbb{R}^3$ to ambient coordinates $\mathbf{X}$ via $\mathbf{x} \mapsto \mathbf{X} = A\mathbf{x} + \mathbf{a}$. A simultaneous change of ambient and body coordinate frames, $\rho = (R,\mathbf{r})$, transforms $\alpha \in SE(3)$ by *conjugation*:

$$\alpha \mapsto \rho \circ \alpha \circ \rho^{-1} = (RAR^{-1}, -RAR^{-1}\mathbf{r} + R\mathbf{a} + \mathbf{r}). \tag{1}$$

Instantaneous displacement at $\alpha \in SE(3)$ is described by the tangent space to the group, in particular that at the identity, its *Lie algebra* $\mathfrak{se}(3)$. An element $\mathbf{s} \in \mathfrak{se}(3)$ is a *twist* and can be represented by a pair of 3-vectors $(\boldsymbol{\omega},\mathbf{v})$, its *twist coordinates*. The vector $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)^T$ can be identified in a standard way with a skew-symmetric 3×3 matrix Ω, such that for $\mathbf{x} \in \mathbb{R}^3$, $\Omega\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$. It can be convenient to represent $\mathbf{s}$ by $(\Omega,\mathbf{v})$. This defines a vector field $\dot{\mathbf{x}} = \Omega\mathbf{x} + \mathbf{v}$ on $\mathbb{R}^3$, whose solution, with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, is $\exp(t\mathbf{s})\mathbf{x}_0$ in $SE(3)$. These are the displacements generated by 1-dof joints, giving the correspondence between joints and twists. The motion is the same for any non-zero multiple of $\mathbf{s}$ so joints are often identified with *screws*, elements of the projective Lie algebra $\mathbb{P}\mathfrak{se}(3)$.

Differentiating (1) gives the *adjoint action* Ad of $SE(3)$ on $\mathfrak{se}(3)$—-how infinitesimal displacements transform under change of coordinates $\rho \in SE(3)$:

$$\mathrm{Ad}\,\rho(\Omega,\mathbf{v}) = (R\Omega R^{-1}, -R\Omega R^{-1}\mathbf{r} + R\mathbf{v}). \tag{2}$$

Differentiating again, with respect to ρ, and evaluating at the identity gives the *adjoint representation* of the Lie algebra. For $\mathbf{s}_i = (\Omega_i,\mathbf{v}_i)$, $i = 1,2$, $\mathrm{ad}\,\mathbf{s}_1(\mathbf{s}_2) = (\Omega_1\Omega_2 - \Omega_2\Omega_1, \Omega_1\mathbf{v}_2 - \Omega_2\mathbf{v}_1)$. We also write $\mathrm{ad}\,\mathbf{s}_1(\mathbf{s}_2) = [\mathbf{s}_1,\mathbf{s}_2]$, the *Lie bracket* of $\mathfrak{se}(3)$, given in twist coordinates $\mathbf{s}_i = (\boldsymbol{\omega}_i,\mathbf{v}_i)$ by $[\mathbf{s}_1,\mathbf{s}_2] = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2)$.

Euclidean displacements can also be represented by dual quaternions [3]. Briefly, a dual quaternion $\check{q} = q_0 + \varepsilon q_1$ has conjugate $\check{q}^* = q_0^* + \varepsilon q_1^*$ (where q^* is the quaternion conjugate of q) and is *unit* if $\check{q}\check{q}^* = 1$. The unit dual quaternions, $\mathbb{D}S^3$, form a 6-dimensional Lie group. The Lie algebra of $\mathbb{D}S^3$ is the set of elements $\check{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}$, where $\boldsymbol{\omega},\mathbf{v}$ are pure imaginary quaternions and the Lie bracket in dual quaternion form is the dualised vector product, $[\check{\boldsymbol{\omega}}_1,\check{\boldsymbol{\omega}}_2] = (\boldsymbol{\omega}_1 + \varepsilon\mathbf{v}_1) \times (\boldsymbol{\omega}_2 + \varepsilon\mathbf{v}_2) = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 + \varepsilon(\boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2)$. There is a 2:1 homomorphism $\theta : \mathbb{D}S^3 \to SE(3)$, $\theta(\check{q}).\mathbf{v} = q_0\mathbf{v}q_0^* + (q_1q_0^* - q_0q_1^*)$, which determines a Lie algebra isomorphism and the adjoint actions of the two groups are equivalent.

The adjoint action of $SE(3)$ induces several other actions of interest. A *multi-twist* is a k-tuple of twists, $(\mathbf{s}_1,\ldots,\mathbf{s}_k) \in \mathfrak{se}(3)^k$ and there is a simultaneous action by $\rho \in SE(3)$:

$$\rho \cdot (\mathbf{s}_1,\ldots,\mathbf{s}_k) = (\mathrm{Ad}\,\rho(\mathbf{s}_1),\ldots,\mathrm{Ad}\,\rho(\mathbf{s}_k)). \tag{3}$$

A manipulator or robot arm with several degrees of freedom, as we shall see in Section 5, is associated with multi-twists. The manipulator's instantaneous capability is a subspace of the Lie algebra $\mathfrak{se}(3)$, i.e. a *screw system of order k*, or k-system [9, 10]. These form a *Grassmannian manifold* $G(k,\mathfrak{se}(3))$ for fixed k and since the adjoint action is linear, there is an induced action of $SE(3)$ on it. Given a k-system $S \in G(k,\mathfrak{se}(3))$ $(1 \le k \le 6)$ and $\mathbf{s}_1,\ldots,\mathbf{s}_k$ a basis for S, there is a well-defined one-to-one function into the projective exterior product (see, for example, [12]):

$$\pi : G(k,\mathfrak{se}(3)) \to \mathbb{P}\left(\wedge^k\mathfrak{se}(3)\right), \quad S \mapsto [\mathbf{s}_1 \wedge \cdots \wedge \mathbf{s}_k], \tag{4}$$

and $\rho \cdot [\mathbf{s}_1 \wedge \cdots \wedge \mathbf{s}_k] = [\mathrm{Ad}\,\rho(\mathbf{s}_1) \wedge \cdots \wedge \mathrm{Ad}\,\rho(\mathbf{s}_k)]$. Let $\mathbf{e}_1,\ldots,\mathbf{e}_6$ denote the standard basis for $\mathfrak{se}(3)$, then $\mathbf{s}_1 \wedge \cdots \wedge \mathbf{s}_k = \sum p_{i_1\cdots i_k}\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$, where the sum is over sequences $i_1,\ldots,i_k$, $1 \le i_1 < \cdots < i_k \le 6$. The coefficients $p_{i_1\cdots i_k}$ may be written in terms of twist coordinates for $\mathbf{s}_i$ and are called *Plücker coordinates* for the exterior product. They satisfy algebraic relations that define the Grassmannian $G(k,\mathfrak{se}(3))$ as a real algebraic variety and a manifold of dimension $k(6-k)$.

3 Invariants of Multi-Twists

The orbits of group actions are important, both mathematically and practically, and invariant theory is a key tool in identifying them. Suppose V is a finite-dimensional real vector (or projective) space and G a group acting on V by linear transformations. If $x_1,\ldots,x_n$ are coordinates for V, the polynomial functions on V form an algebra $\mathbb{R}[x_1,\ldots,x_n]$, abbreviated as $\mathbb{R}[\mathbf{x}]$. There is an induced action of G on $\mathbb{R}[\mathbf{x}]$: for $g \in G$, $f \in \mathbb{R}[\mathbf{x}]$ and $\mathbf{x} \in V$, $(g \cdot f)(\mathbf{x}) = f(g \cdot \mathbf{x})$. An *invariant polynomial* is an element $f \in \mathbb{R}[\mathbf{x}]$ such that for all $g \in G$, $g \cdot f = f$, and they form a subalgebra $\mathbb{R}[\mathbf{x}]^G$. For many groups G, e.g. finite, semisimple or reductive , $\mathbb{R}[\mathbf{x}]^G$ is *finitely generated* [14]. This is called the First Fundamental Theorem of Invariant Theory, in respect of G. There exist non-reductive groups for which finite generation does not hold and the Euclidean group is not reductive, though we know invariant rings are finitely generated for some representations. For the adjoint action of $SE(3)$, the polynomial ring is $\mathbb{R}[\omega_1,\omega_2,\omega_3,v_1,v_2,v_3]$, (or $\mathbb{R}[\boldsymbol{\omega},\mathbf{v}]$). In the case $n=3$ of a theorem [8] for $SE(n)$ we have:

Theorem 1. *The invariant ring* $\mathbb{R}[\boldsymbol{\omega},\mathbf{v}]^{SE(3)}$ *is the polynomial ring* $\mathbb{R}[\boldsymbol{\omega}.\boldsymbol{\omega},\boldsymbol{\omega}.\mathbf{v}]$.

The generating invariants are multiples of the Killing and Klein forms, $\boldsymbol{\omega}.\boldsymbol{\omega} = -\frac{1}{2}\mathbf{s}^T A\mathbf{s}$, $\boldsymbol{\omega}.\mathbf{v} = \frac{1}{2}\mathbf{s}^T B\mathbf{s}$, where $A = \begin{pmatrix} -2I_3 & O_3 \\ O_3 & O_3 \end{pmatrix}$, $B = \begin{pmatrix} O_3 & I_3 \\ I_3 & O_3 \end{pmatrix}$.

The forms $Q_{\alpha,\beta} = \alpha A + \beta B$, $\alpha, \beta \in \mathbb{R}$ are non-degenerate if and only if $\beta \neq 0$ and then are indefinite of index 3. The equations $q_{\alpha,\beta}(\mathbf{s}) = \mathbf{s}^T Q_{\alpha,\beta}\mathbf{s} = 0$ define a family of hypersurfaces, the *pitch quadrics*, in the screw space $\mathbb{P}\mathfrak{se}(3)$. For $\beta \neq 0$, these are parametrised by the invariant *pitch of the screw* $h = \alpha/\beta = \boldsymbol{\omega}.\mathbf{v}/\boldsymbol{\omega}.\boldsymbol{\omega}$, and we write $q_h = 0$. For $\beta = 0$, set $h = \infty$ and the variety $q_\infty = 0$ is the projective plane $\boldsymbol{\omega} = \mathbf{0}$, which lies in the intersection of all the pitch quadrics.

The Principle of Transference [11, 13] may be used to find invariants of multi-twists by dualising the vector invariants of the rotation group $SO(3)$ [15]. Specifically, given $f \in \mathbb{R}[\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_k]$, its dual form $\check{f}$ is given by replacing its variables by dual quantitites $\boldsymbol{\omega}_i + \varepsilon \mathbf{v}_i$, expanding its Maclaurin series and using $\varepsilon^2 = 0$:

$$\check{f}(\boldsymbol{\omega}_1 + \varepsilon\mathbf{v}_1, \cdots, \boldsymbol{\omega}_k + \varepsilon\mathbf{v}_k) = f(\boldsymbol{\omega}_1, \cdots, \boldsymbol{\omega}_k) + \varepsilon\left(\sum_{r=1}^{k} \mathbf{v}_r \frac{\partial f}{\partial \boldsymbol{\omega}_r}(\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_k)\right). \quad (5)$$

The dual part, $\tilde{f}$, is the partial polarisation of f. For example, the dual of the invariant form $f(\boldsymbol{\omega}) = \boldsymbol{\omega}.\boldsymbol{\omega}$ is $\check{f}(\boldsymbol{\omega}, \mathbf{v}) = \boldsymbol{\omega}.\boldsymbol{\omega} + \varepsilon\, \boldsymbol{\omega}.\mathbf{v}$, giving the Killing and Klein forms, respectively. The following theorem [6] realises multi-twist invariants:

Theorem 2. *If f is a vector invariant of the adjoint action of $SO(3)$, then the primal and dual parts of $\check{f}$ are vector invariants of the adjoint action of $SE(3)$. The primal and dual parts of the dualisation of any $SO(3)$ syzygy are syzygies for $SE(3)$.*

Dualising the generators for vector invariants [15] of $SO(3)$ gives:

1. for $1 \leq i \leq j \leq m$,

$$\check{I}_{ij} := I_{ij} + \varepsilon\tilde{I}_{ij} := \boldsymbol{\omega}_i.\boldsymbol{\omega}_j + \varepsilon(\boldsymbol{\omega}_i.\mathbf{v}_j + \boldsymbol{\omega}_j.\mathbf{v}_i) = -\tfrac{1}{2}\left(\mathbf{s}_i^T Q_\infty \mathbf{s}_j + \varepsilon\mathbf{s}_i^T Q_0 \mathbf{s}_j\right); \quad (6)$$

2. for $m \geq 3$ and any $1 \leq i < j < k \leq m$, setting $[\mathbf{u}\;\mathbf{v}\;\mathbf{w}] = \mathbf{u}.(\mathbf{v} \times \mathbf{w})$,

$$\begin{aligned}\check{I}_{ijk} := I_{ijk} + \varepsilon\tilde{I}_{ijk} : &= [\boldsymbol{\omega}_i\; \boldsymbol{\omega}_j\; \boldsymbol{\omega}_k] + \varepsilon\left([\boldsymbol{\omega}_i\; \boldsymbol{\omega}_j\; \mathbf{v}_k] + [\boldsymbol{\omega}_i\; \mathbf{v}_j\; \boldsymbol{\omega}_k] + [\mathbf{v}_i\; \boldsymbol{\omega}_j\; \boldsymbol{\omega}_k]\right) \\ &= -\tfrac{1}{2}\left(\mathbf{s}_i^T Q_\infty[\mathbf{s}_j, \mathbf{s}_k] + \varepsilon\mathbf{s}_i^T Q_0[\mathbf{s}_j, \mathbf{s}_k]\right), \end{aligned} \quad (7)$$

For $m = 2$, there are 6 (quadratic) invariants and they generate the ring of invariants. For $m = 3$, there are 14 invariants, but is not known whether the First Fundamental Theorem holds for $m \geq 3$. However, the theorem in [6] that, for $m = 3$, every polynomial invariant can be written as a rational expression in $I_{ij}, \tilde{I}_{ij}$ and I_{123}, extends to the general case $m > 3$, still with only the one cubic invariant. The invariants are connected by three types of dualised syzygy [15], which we do not specify here.

4 Invariants of screw systems

Hunt's classification [10] of Ball's screw systems is given a mathematical basis by Gibson and Hunt (GH) [9], where a normal form of basis twists for each screw system is given. Classes are unions of orbits and the normal forms include invariant parameters (moduli). For a generic 2-system (GH-type IA) the moduli h_1, h_2 are the principal pitches, which can be found by solving the discriminant $\det(\mathbf{S}^T Q_h \mathbf{S}) = 0$, where $\mathbf{S}$ is the 6×2 matrix whose columns are spanning twists. Type IB systems have a unique infinite pitch screw and contain a unique screw of every pitch whose axes are coplanar. The normal form has a modulus q, which is the tangent of the angle this plane makes with the infinitesimal translation. The generic 3-system, type IA, has normal form with moduli the three principal pitches $h_1 < h_2 < h_3$ corresponding to the degenerate or singular intersections with the pitch quadrics, which can be found analogously with 2-systems.

Selig [12] explores the invariant theory for screw systems along these lines. For a 2-system $S = [\mathbf{s}_1 \wedge \mathbf{s}_2]$, the coefficients $\iota_1, \iota_2, \iota_3$ of $\det \mathbf{S}^T Q_{\alpha,\beta} \mathbf{S} = \iota_1 \beta^2 + \iota_2 \alpha\beta + \iota_3 \alpha^2$ are invariants of the twist-pair $\mathbf{s}_1, \mathbf{s}_2$. Further, if M is a change of basis for the 2-system then $\iota_r' = (\det M)^2 \iota_r$ for $r = 1, 2, 3$, so they are screw system invariants. They are sufficient to distinguish most, but not all, of the 2-system orbits. For type IA, the principal pitches h_1, h_2 are the solutions of the quadratic $\iota_3 h^2 + \iota_2 h + \iota_1 = 0$, giving $h_1 + h_2 = -\iota_2/\iota_3$, $h_1 h_2 = \iota_1/\iota_3$. The modulus q for IB systems, which separates orbits, cannot be found from $\iota_1, \iota_2, \iota_3$ since these invariants vanish for all q. We show below that these are the only invariants. (The expressions i_4, i_5 in [12], Section 8.4 are not true invariants.) Invariants for 3-systems are also identified in [12].

We adapt this approach using the invariant ring for k twists. Form the quadratic vector invariants into $k \times k$ symmetric matrices $\mathbf{I} = (I_{ij})$, $\tilde{\mathbf{I}} = (\tilde{I}_{ij})$. A change of basis $M = (m_{ij}) \in GL(k)$ transforms $\mathbf{S} = (\mathbf{s}_1, \ldots, \mathbf{s}_k) \mapsto \mathbf{S}_i' = \mathbf{S}M$ and, by (6), the invariants transform in the following way:

$$I_{ij}' = -\tfrac{1}{2}\mathbf{s}_i' Q_\infty \mathbf{s}_j' = -\tfrac{1}{2}\sum_r \sum_s m_{ri} m_{sj} (\mathbf{s}_r^T Q_\infty \mathbf{s}_j) = -\tfrac{1}{2}\sum_r \sum_s m_{ri} m_{sj} I_{rs} = (M^T \mathbf{I} M)_{ij} \tag{8}$$

and similarly for $\tilde{I}_{ij}'$, replacing Q_∞ by Q_0. Thus, the action of $GL(k)$ on the quadratic k-twist invariants is the same as its action on a pair of k-ary quadratic forms. In the case $k = 2$, the invariant ring is:

$$\mathbb{R}[\boldsymbol{\omega}_1, \mathbf{v}_1, \boldsymbol{\omega}_2, \mathbf{v}_2]^{SE(3)} = \mathbb{R}[I_{11}, I_{12}, I_{22}, \tilde{I}_{11}, \tilde{I}_{12}, \tilde{I}_{22}], \tag{9}$$

and the 6×6 representation of $M \in GL(2)$ is the double symmetric square of M:

$$\begin{pmatrix} \tilde{M} & O \\ O & \tilde{M} \end{pmatrix} \quad \text{where} \quad \tilde{M} = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{21} & m_{21}^2 \\ m_{11}m_{12} & m_{11}m_{22} + m_{12}m_{21} & m_{21}m_{22} \\ m_{12}^2 & 2m_{12}m_{22} & m_{22}^2 \end{pmatrix}. \tag{10}$$

From classical invariant theory [14], we have the following invariants:

$$j_1 = \tilde{I}_{11}\tilde{I}_{22} - \tilde{I}_{12}^2, \quad j_2 = I_{11}\tilde{I}_{22} + I_{22}\tilde{I}_{11} - 2I_{12}\tilde{I}_{12}, \quad j_3 = I_{11}I_{22} - I_{12}^2. \tag{11}$$

A simple way to view these is as the coefficients of $\det(\alpha\mathbf{I} + \beta\tilde{\mathbf{I}})$. Substituting the Plücker coordinate expressions for the 2-twist invariants and for the coordinates p_{ij}

confirms that $j_r = i_r$, $r = 1,2,3$ (up to non-zero multiples). By (9), this argument shows that j_1, j_2, j_3 form a generating set for the invariants of 2-systems.

For $k = 3$, in addition to 12 quadratic 3-twist invariants there are two cubic invariants. Since $I_{123} = \boldsymbol{\omega}_1.(\boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3)$ is the volume of the parallelepiped spanned by these vectors, it satisfies for $M \in GL(3)$: $I'_{123} = (\det M) I_{123}$. Polarising this equation shows that $\tilde{I}_{123}$ is also an invariant. Now consider the action of $GL(3)$ on the polynomial ring generated by I_{ij}, $\tilde{I}_{ij}$ that corresponds to the action on a pair of ternary quadratics. A set of generating invariants is the coefficients of $\det(\alpha\mathbf{I} + \beta\tilde{\mathbf{I}})$:

$$\begin{aligned}
k_1 &= \tilde{I}_{11}\tilde{I}_{22}\tilde{I}_{33} - \tilde{I}_{11}\tilde{I}_{23}^2 - \tilde{I}_{22}\tilde{I}_{13}^2 - \tilde{I}_{33}\tilde{I}_{12}^2 + 2\tilde{I}_{12}\tilde{I}_{13}\tilde{I}_{23} \\
k_2 &= I_{11}\tilde{I}_{22}\tilde{I}_{33} + I_{22}\tilde{I}_{11}\tilde{I}_{33} + I_{33}\tilde{I}_{11}\tilde{I}_{22} - I_{11}\tilde{I}_{23}^2 - I_{22}\tilde{I}_{13}^2 - I_{33}\tilde{I}_{12}^2 \\
&\quad + 2I_{12}(\tilde{I}_{13}\tilde{I}_{23} - \tilde{I}_{12}\tilde{I}_{33}) + 2I_{13}(\tilde{I}_{12}\tilde{I}_{23} - \tilde{I}_{13}\tilde{I}_{22}) + 2I_{23}(\tilde{I}_{12}\tilde{I}_{13} - \tilde{I}_{11}\tilde{I}_{23}) \\
k_3 &= I_{11}I_{22}\tilde{I}_{33} + I_{11}I_{33}\tilde{I}_{22} + I_{22}I_{33}\tilde{I}_{11} - I_{12}^2\tilde{I}_{33} - I_{13}^2\tilde{I}_{22} - I_{23}^2\tilde{I}_{11} \\
&\quad + 2(I_{13}I_{23} - I_{12}I_{33})\tilde{I}_{12} + 2(I_{12}I_{23} - I_{13}I_{22})\tilde{I}_{13} + 2(I_{12}I_{13} - I_{11}I_{23})\tilde{I}_{23} \\
k_4 &= I_{11}I_{22}I_{33} - I_{11}I_{23}^2 - I_{22}I_{13}^2 - I_{33}I_{12}^2 + 2I_{12}I_{13}I_{23}.
\end{aligned} \tag{12}$$

These satisfy the identities $k_4 = I_{123}^2, k_3 = 2I_{123}\tilde{I}_{123}$, being the two syzygies for 3-twist invariants [6]. If, in fact, (6,7) generate 3-twist invariants, then $I_{123}, \tilde{I}_{123}, k_1, k_2$ generate the invariant polynomials for 3-systems, however this remains a conjecture. The invariants correspond to Selig's: $\sqrt{-\frac{1}{2}\det \Upsilon_\infty}$, i_0, $\det \Upsilon_0$, $\boldsymbol{\Phi}$, respectively. As for 2-systems, these invariants determine the principal pitches.

5 Invariants and Serial Manipulators

Brockett's product of exponentials formulation for a m-dof serial manipulator [1] requires the choice of m twists, $\mathbf{s}_1, \ldots, \mathbf{s}_m$, representing the joints in the home configuration. In principle, they may have any pitch, but in practice are usually pitch zero or infinity, i.e. revolute or prismatic. The forward kinematics for the end-effector is $f : \mathbb{R}^m \to SE(3)$, $f(u_1, \ldots, u_m) = \exp(u_1\mathbf{s}_1)\exp(u_2\mathbf{s}_2)\cdots\exp(u_m\mathbf{s}_m)$, where $u_1, \ldots, u_m$ are joint variables. As the arm moves, so do the joints (other than $\mathbf{s}_1$) and the multi-twist changes. At configuration $\mathbf{u} = (u_1, \ldots, u_m)$:

$$\mathbf{s}_2(\mathbf{u}) = \mathrm{Ad}\,(\exp(u_1\mathbf{s}_1))\mathbf{s}_2, \ldots, \mathbf{s}_m(\mathbf{u}) = \mathrm{Ad}\,(\exp(u_1\mathbf{s}_1)\cdots\exp(u_{m-1}\mathbf{s}_{m-1}))\mathbf{s}_m. \tag{13}$$

From the point of view of invariants, in addition to the joint adjoint action on multi-twists, we must also consider the internal action of earlier joints on later joints. Notice that motion around the first joint alone gives an equivalent multi-twist, since we may apply $\mathrm{Ad}\,(\exp(-u_1\mathbf{s}_1))$ to the updated twists in (13) and $\mathbf{s}_1$ is fixed by this. Also, clearly motion about the final joint has no effect.

An alternative formalism for serial kinematics is Denavit–Hartenberg (DH) parameters [7]. A coordinate frame is chosen for each link in such a way that the joint twists have a standard form. The exponential motions generated by each joint must be connected by corresponding changes of coordinates, expressed in terms of the DH parameters: *link length*, *link angle* and *offset*. It is implicitly recognised that these parameters are invariant under global coordinate change (the adjoint action on the multi-twist), which we refer to as *static invariance*, and the internal motion of the manipulator or *kinematic invariance*. The connection between the static, multi-twist, invariants and the DH parameters is established in [4,5]. Explicit formulae are obtained for each in terms of the invariants (6,7). For a multi-twist $(\mathbf{s}_1,\ldots,\mathbf{s}_m)$,

- invariants $I_{ii},\tilde{I}_{ii}$, $i=1,\ldots,m$ tell us about the individual joints and their pitch h_i.
- for each pair $\mathbf{s}_i,\mathbf{s}_j$, $1\le i<j\le m$, the invariants $I_{ij},\tilde{I}_{ij}$ determine the link length l_{ij}—the perpendicular distance between the axes of the twists—and link angle θ_{ij}—the angle between these axes, where these are defined.
- for each triple $\mathbf{s}_i,\mathbf{s}_j,\mathbf{s}_k$, $1\le i<j<k\le m$, the quadratic plus cubic invariants $I_{ijk},\tilde{I}_{ijk}$, determine the offsets d_{ijk}, the distance along the axis of $\mathbf{s}_j$ between the feet of the perpendiculars from $\mathbf{s}_i,\mathbf{s}_k$, again so long as this is well defined.

Explicit formulae for the screw and DH parameters are:

$$h_i=\frac{\tilde{I}_{ii}}{I_{ii}},\; l_{ij}=\frac{I_{ij}(I_{ii}\tilde{I}_{jj}+\tilde{I}_{ii}I_{jj})-I_{ii}I_{jj}\tilde{I}_{ij}}{I_{ii}I_{jj}\sqrt{I_{ii}I_{jj}-I_{ij}^2}},\;\cos\theta_{ij}=\frac{I_{ij}}{\sqrt{I_{ii}I_{jj}}},\; l_{ij}\sin\theta_{ij}=\frac{\tilde{I}_{ij}}{\sqrt{I_{ii}I_{jj}}}$$

$$d_{ijk}=\frac{\tilde{I}_{ijk}I_{jj}(I_{ij}I_{jk}-I_{ik}I_{jj})+I_{ijk}\left(\frac{1}{2}\tilde{I}_{jj}(I_{ij}I_{jk}+I_{ik}I_{jj})+I_{jj}(\tilde{I}_{ik}I_{jj}-I_{ij}\tilde{I}_{jk}-\tilde{I}_{ij}I_{jk})\right)}{\sqrt{I_{jj}}(I_{ii}I_{jj}-I_{ij}^2)(I_{jj}I_{kk}-I_{jk}^2)}. \tag{14}$$

We have $I_{ii}=0$ if and only if $\boldsymbol{\omega}_i=0$, so that $\mathbf{s}_i$ is prismatic. The invariant expression $I_{ii}I_{jj}-I_{ij}^2=\|\boldsymbol{\omega}_i\times\boldsymbol{\omega}_j\|^2$, so vanishes if $\boldsymbol{\omega}_i=0$, $\boldsymbol{\omega}_j=0$ (i.e. $\mathbf{s}_i$ or $\mathbf{s}_j$ prismatic) or $\boldsymbol{\omega}_i\|\boldsymbol{\omega}_j$—so no unique common perpendicular between the joint axes.

Verifying that an expression is invariant with respect to the action of a (connected) Lie group is equivalent to showing that it is invariant under the infinitesimal action of the Lie algebra, found by differentiating at the identity. Specifically, since the derivative of the Euclidean group's adjoint action is the Lie bracket, if $f(\mathbf{s}_i,\ldots,\mathbf{s}_m)$ is a multi-twist invariant, then it is an invariant of the motion about joint r, $2\le r\le m-1$, of a serial manipulator if and only if:

$$\nabla f(\mathbf{s}_1,\ldots,\mathbf{s}_m)\cdot(\mathbf{0},\ldots,\mathbf{0},[\mathbf{s}_r,\mathbf{s}_{r+1}],\ldots,[\mathbf{s}_r,\mathbf{s}_m])=0. \tag{15}$$

This is the $6m$-dimensional gradient vector multiplied by the column consisting of r zero 6-vectors, followed by $m-r$ brackets in the Lie algebra $\mathfrak{se}(3)$.

Theorem 3. *For a serial manipulator with joint twists $\mathbf{s}_1,\ldots,\mathbf{s}_m$, the only static invariants that are kinematically invariant are $I_{ii},\tilde{I}_{ii}$, $1\le i\le m$ and $I_{i,i+1},\tilde{I}_{i,i+1}$, $1\le i\le m-1$.*

In particular, the invariants $I_{ik},\tilde{I}_{ik}$, $k\ge i+2$ are not kinematically invariant under the action of joint j for $i<j<k$. The cubic invariants $I_{ijk},\tilde{I}_{ijk}$ are not generally kinematically invariant for any i,j,k. It follows from (14) that pitches of joints and link

lengths and angles for successive joints are kinematically invariant, while offsets are not. However, we have the following special case.

Theorem 4. *For a serial manipulator as in Thm. 3, the invariants $I_{i,i+1,i+2}, \tilde{I}_{i,i+1,i+2}$ for $1 \leq i \leq m-2$ are kinematically invariant so long as $\tilde{I}_{i+1} = 0$. Thus, the offset for 3 successive joints is invariant if the middle joint is revolute.*

6 Conclusion

We have shown that the invariant polynomials of multi-twists play a valuable role in understanding the screw systems and the kinematics of serial manipulators. They may throw light on other aspects of manipulator kinematics, such as *persistence* [2] in which the screw system remains constant under motion of the manipulator's own motion. Given their algebraic simplicity, they provide an alternative to DH parameters that are well-adapted to product-of-exponentials formalism.

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