Metrical problems in Diophantine approximation

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Abstract

The fundamental result in the theory of metric Diophantine approximation is Dirichlet's Theorem (1842) that gives an error of approximation for all irrational numbers by rationals. In the literature this theorem is also referred to as the *uniform Diophantine approximation result.* A weaker form of this theorem sometimes known as the *asymptotic Diophantine approximation result*, is that there are infinitely many integer solutions for any irrational number with an error of approximation one over the denominator squared. Most of the developments to date, such as the classical Khintchine (1924), Jarník–Besicovitch (1928, 1934) and Jarník (1931) theorems are concerned with the strengthening and generalisations of the asymptotic version of Dirichlet's Theorem rather than Dirichlet's Theorem itself. In this thesis, by building on recent results of Kleinbock–Wadleigh (2018) we present a nearly complete metrical description of the sets of Dirichlet non-improvable numbers.

Statement of Authorship

This thesis includes work by the author that has been published or submitted as described in the text. Except where reference is made in the text of the thesis, this thesis contains no other material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

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An overview of thesis

The thesis will proceed along the following lines.

In Chapter 1, we provide an introduction to the classical theory of Diophantine approximation that will form the background of the thesis.

Chapter 2 is reserved for the auxiliary results and definitions. Some elementary properties of continued fractions are also discussed that we will use in the sequel. Specifically details are given for the pressure functions related with the Gauss dynamical system and how solutions to such pressure functions give the Hausdorff dimension of the corresponding sets.

In Chapter 3, we present the Lebesgue measure theory for the set of Dirichlet non-improvable numbers as well as its Hausdorff measure analogue which are due to Kleinbock–Wadleigh [34] and Hussain–Kleinbock–Wadleigh–Wang [27] respectively.

In Chapter 4, we present our first result. To state the result we introduce some notation first. Let $\Psi : [1, \infty) \to \mathbb{R}^+$ be a non-decreasing function, $a_n(x)$ the n-th partial quotient of x and $q_n(x)$ the denominator of the n-th convergent. The set

$$G(\Psi) := \Big\{ x \in [0,1) : a_n(x)a_{n+1}(x) \; > \; \Psi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N} \Big\},$$

of Dirichlet non-improvable numbers is related with the classical set of $\frac{1}{q\Psi(q)}$-well approximable numbers $\mathcal{K}(\Psi)$ in the sense that $\mathcal{K}(3\Psi) \subset G(\Psi)$. Both of these sets enjoy the same s-dimensional Hausdorff measure criterion for $s \in (0,1)$. We prove that the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$ is uncountable by proving that it has the same Hausdorff dimension as that for the sets $\mathcal{K}(\Psi)$ and $G(\Psi)$. This gives an affirmative answer to a question raised by Hussain–Kleinbock–Wadleigh–Wang [27].

In Chapter 5, we calculate the Hausdorff dimension of the set

$$\mathcal{F}(\Phi) := \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\},$$

where $\Phi : \mathbb{N} \to (1, \infty)$ is any function with $\lim_{n\to\infty} \Phi(n) = \infty$. This in turn contributes to the metrical theory of continued fractions and also gives insights about the set of Dirichlet non-improvable numbers.

In Chapter 6, for any $r \in \mathbb{N}$, we investigate the Hausdorff dimension of the following set

$$\mathcal{R}_r(\tau; h) := \left\{ x \in [0,1) : \prod_{d=1}^{r} a_{n+d}(x) \geq e^{\tau(x)(S_n h(x))} \text{ for infinitely many } n \in \mathbb{N} \right\},$$

where h and τ are positive continuous functions, $S_n h(x) := h(x) + \cdots + h(T^{n-1}(x))$ is the ergodic sum and T represents the Gauss map.

Chapter 7 summarises the results of this thesis discussed in previous chapters as well as including some recent results arising in the one-dimensional uniform approximation theory concerned with the metrical theory of the sets of Dirichlet non-improvable numbers.

Chapter 1

Introduction

In this chapter, we provide an introduction to the classical theory of Diophantine approximation that will form the background of the thesis. Therein we start from the basic notions. Throughout this chapter we state classical results from [6, 11, 31].

1.1 Diophantine approximation

The elementary objective of the theory of Diophantine approximation is to seek an answer to the question *'How rapidly can an irrational number be approximated by a sequence of rational numbers?'*.

There are two approaches to approximate a real number by rational numbers: the *qualitative approach* and the *quantitative approach*.

A qualitative approach follows from the fact that the rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$. To be more precise, given any real number $x \in \mathbb{R}$ we can always construct a sequence of rational numbers $\{r_n\}_{n\geq 1}$ such that $r_n \to x$ as $n \to \infty$. That is, we can always find a rational in a δ-neighbourhood of x for any $\delta > 0$.

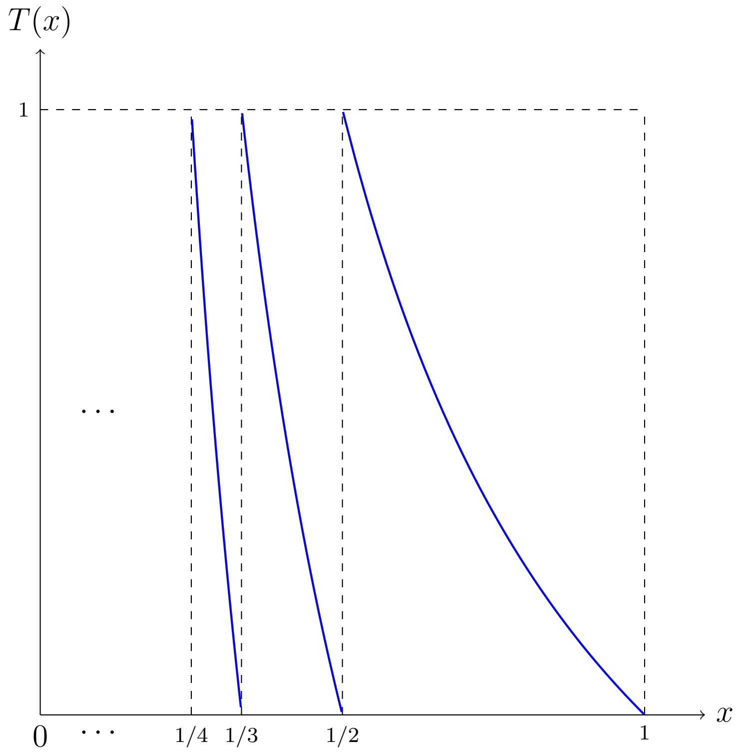
Studying the quantitative approach leads to the theory of metric Diophantine approximation. The theory quantifies the closeness (approximation) of irrational numbers by rational numbers. If the denominator of a rational number $\frac{p}{q}$, where $q > 0$ is fixed, then since the distance between two rationals with same denominator q is exactly $\frac{1}{q}$, every real number can be approximated by a rational number with error of approximation not exceeding $\frac{1}{2q}$.

Continued fraction expansions are considered as one of the important tools to study problems in Diophantine approximation. The theory of continued fractions provides a quick and efficient way of finding good rational approximations to irrational numbers.

Every irrational number admits a unique infinite continued fraction expansion. We will start by defining a map $T : [0, 1) \to [0, 1)$ known as the Gauss map. The Gauss map plays the role of operator in the continued fraction algorithm and is defined as

$$T(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ \\ \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if} \quad 0 < x < 1, \end{cases} \tag{1.1}$$

where $[\star]$ denotes the integer part. Each branch T is monotone, surjective and invertible (see Figure 'The Gauss map' below).



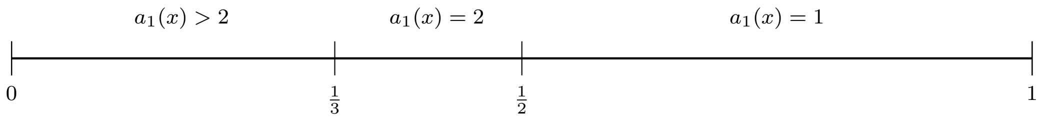
The Gauss map.

For each $n \geq 1$, the n-th *partial quotient* '$a_n(x)$' of x is defined as

$$a_n(x) := \left[\frac{1}{T^{n-1}(x)}\right] \in \mathbb{N}.$$

As we are considering $a_0(x) = 0$ and we get a unique continued fraction expansion

$$\begin{aligned} x &= \frac{1}{a_1(x) + T^1 x} = \frac{1}{a_1(x) + \cfrac{1}{a_2(x) + T^2 x}} \\ &= \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ldots \frac{1}{a_n(x) + T^n x}}}} \\ &:= [a_1(x), a_2(x), \ldots, a_n(x) + T^n x]. \end{aligned}$$



The location of the first partial quotient in the unit interval.

The existence of the limit

$$x = [a_1(x), a_2(x), \ldots, a_n(x), \ldots] = \lim_{n\to\infty} [a_1(x), a_2(x), \ldots, a_n(x) + T^n x]$$

is due to the *principal convergents* '$\frac{p_n(x)}{q_n(x)}$' that are obtained by truncating the continued fraction expansion of x at level n.

Throughout the thesis, just for simplicity, sometimes we may write p_n, q_n, a_n for $p_n(x)$, $q_n(x)$, $a_n(x)$ respectively.

These principal convergents are defined by setting $p_{-1} := 1$, $q_{-1} := 0$, $p_0 := 0$, $q_0 := 1$ and then followed by the recursively defined formulas

$$p_n = a_n(x)p_{n-1} + p_{n-2}, \quad \text{and} \quad q_n = a_n(x)q_{n-1} + q_{n-2} \quad \text{for} \quad n \in \mathbb{N}. \tag{1.2}$$

Also,

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n, \text{ for all } n \geq 1. \tag{1.3}$$

These convergents satisfy the inequalities

$$\frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

For example, let's have a look at convergents of π. A short computation shows that the first few convergents of π are located as follows:

$$\frac{3}{1} < \frac{333}{106} < \frac{103993}{33102} < \cdots < \pi < \cdots < \frac{355}{113} < \frac{22}{7}.$$

In fact, these rationals are considered as the best approximations to π. Thus continued fractions help in providing good rational approximations to irrational numbers. In fact, this observation is due to an important result by Lagrange (1770) that identifies the convergents in the continued fraction representation of an irrational x with the sequence of 'good approximations.' According to Lagrange's Theorem, if $x \in [0, 1)$ is an irrational number with a sequence of convergents $\left\{\frac{p_n}{q_n}\right\}_{n\geq 1}$ in its continued fraction expansion, then

$$|q_1x - p_1| > |q_2x - p_2| > \cdots > |q_nx - p_n| > \cdots.$$

Further, if $n \geq 1$, and p, q are integers with $\frac{p}{q} \neq \frac{p_n}{q_n}$ and $1 \leq q \leq q_n$. Then

$$|qx - p| > |q_nx - p_n|.$$

Thus the above result by Lagrange, sometimes known as 'Law of best approximation' reveals that we cannot do better than approximating an irrational by its convergents.

Whereas according to Legendre's Theorem if p, q are relatively prime integers with $q \geq 1$ and $\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent to x, i.e.,

$$\frac{p}{q} = \frac{p_n}{q_n} \text{ for some } n. \tag{1.4}$$

Thus we need to focus on convergents in order to find good approximations to x. In fact, for

$$x = \frac{p_{n+1} + T^{n+1}(x)p_n}{q_{n+1} + T^{n+1}(x)q_n},$$

we have

$$\frac{1}{3a_{n+1}(x)q_n^2} < \left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n(q_n T^{n+1}(x) + q_{n+1})} < \frac{1}{a_{n+1}(x)q_n^2}. \tag{1.5}$$

From (1.5) it is obvious that we obtain good rational approximations when the partial quotient $a_{n+1}(x)$ is unbounded. Thus we can note that the Diophantine properties of x can be determined by the growth rate of its partial quotients. In view of (1.5) as n tends to infinity the sequence $\frac{p_n}{q_n}$ of convergents is approximating x better and better and the increase in the accuracy of $\frac{p_n}{q_n}$ upon the previous convergents is proportional to the next partial quotient $a_{n+1}(x)$. Thus producing infinitely many solutions $\frac{p_n}{q_n}$ which satisfies (1.5). Together with the fact that the partial quotients $a_n(x)$ are always greater than or equal to one for each $n \geq 1$ and from (1.5) it follows that.

Theorem 1.1.1 *For an irrational x there exists infinitely many rationals $\frac{p}{q}$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{1.6}$$

Thus by (1.5) the convergents to an irrational number x satisfy (1.6).

Returning to the trivial bound $\frac{1}{2q}$, where q is fixed, the natural question was can we improve this bound? There are various approaches to improving this bound. The key approach in this direction is due to Dirichlet (1805-1859).

His landmark result, also known as Dirichlet's Theorem, which sits at the heart of Diophantine approximation, is concerned with approximating the real numbers by rationals having *bounded* denominators and is based on counting argument known as the 'Pigeonhole principle'.

Theorem 1.1.2 (Dirichlet, 1842) *Given $x \in \mathbb{R}$ and $t > 1$, there exists $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that*

$$|qx - p| < \frac{1}{t} \quad \text{and} \quad 1 \leq q \leq t. \tag{1.7}$$

Proof: Recall that $[x] := \{n \in \mathbb{Z} : n \leq x\}$ denotes the integer part and $\{x\}$ denotes the fractional part of any $x \in \mathbb{R}$.

Consider the $t+1$ numbers

$$\{0x\}, \{x\}, \{2x\}, ..., \{tx\}.$$

All these fractional parts are contained in the unit interval $[0,1)$ since $0 \leq \{x\} < 1$ for any $x \in \mathbb{R}$. Divide $[0,1)$ into t equal semi-open subintervals as

$$[0,1) = \Big[0, \frac{1}{t}\Big) \cup \Big[\frac{1}{t}, \frac{2}{t}\Big) \cup \cdots \cup \Big[\frac{t-1}{t}, 1\Big). \tag{1.8}$$

Since the $t+1$ fractional parts are situated in t subintervals (1.8), the Pigeonhole principle guarantees that there are two integers $0 \leq q_1 < q_2 \leq t$ such that $\{q_2 x\}$, $\{q_1 x\}$ are contained in same interval. Since the length of each semi-open interval is t^{-1} we have that

$$|\{q_2 x\} - \{q_1 x\}| < \frac{1}{t}.$$

Let $q_i x = p_i + \{q_i x\}$ where $p_i = [q_i x]$ for $1 \leq i \leq 2$. Then

$$|\{q_2 x\} - \{q_1 x\}| = |(q_2 - q_1)x - (p_2 - p_1)| < \frac{1}{t}.$$

Now define $q = q_2 - q_1 \in \mathbb{N}$ and $p = p_2 - p_1 \in \mathbb{Z}$. Since $0 \leq q_1, q_2 \leq t$ and $q_1 < q_2$ we have $1 \leq q \leq t$ such that

$$|qx - p| < \frac{1}{t},$$

which completes the proof. ■

Thus from Dirichlet's Theorem it is obvious that the trivial bound $\frac{1}{2q}$ can be naturally improved upon.

Dirichlet's Theorem is the archetypal *uniform* Diophantine approximation result, so called as it guarantees a non-trivial integer solution for all t. A weaker form guarantees that such a system is solvable for an unbounded set of t, sometimes known as asymptotic approximation – for example [43, 32]. For instance, Dirichlet's Theorem implies that (1.7) is solvable for an unbounded set of t, a fortiori. The following corollary which follows trivially from this weaker statement and is the standard application of (1.7) is an archetypal asymptotic approximation result.

Corollary 1.1.3 *For any irrational $x \in \mathbb{R}$, there exist infinitely many $(p,q) \in \mathbb{Z} \times \mathbb{N}$ such that $\gcd(p,q) = 1$ and*

$$|qx - p| < \frac{1}{q}. \tag{1.9}$$

Proof: Let us begin by assuming x is irrational and there are only finitely many pairs $(p_i, q_i) \in \mathbb{Z} \times \mathbb{N}$ with gcd $(p_i, q_i) = 1$ such that

$$|q_i x - p_i| < \frac{1}{q_i},$$

for each $1 \leq i \leq n$. Since x is irrational the difference $q_i x - p_i$ will be non-zero and therefore there exists $t \in \mathbb{N}$ such that

$$|q_i x - p_i| > \frac{1}{t} \text{ for all } i.$$

By Theorem 1.1.2 there exists a pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $\gcd(p, q) = 1$ such that

$$|qx - p| < \frac{1}{qt} \leq \frac{1}{t} \quad \text{with} \quad 1 \leq q \leq t.$$

Therefore, $(p, q) \neq (p_i, q_i)$ for any i but satisfies (1.9) which is a contradiction. ∎

Thus the two statements above show two possible ways to pose Diophantine approximation problems, as discussed earlier often referred to as uniform vs. asymptotic: that is, looking for solvability of inequalities for all large enough values of certain parameters vs. for infinitely many (a distinction between limsup and liminf sets). The rate of approximation given in (1.7) and (1.9) works for all x, which serves as the beginning of the *metric theory of Diophantine approximation*, a field concerned with understanding sets of x satisfying similar conclusions but with the right hand sides replaced by faster decaying functions of t and q respectively. Those sets are well studied in the asymptotic set-up (1.9) long ago while the analogous questions about the uniform set-up remain open.

We will start by giving an overview of what is known in asymptotic set-up including the landmark results of Khintchine (1924), Jarník–Besicovitch (1928, 1934) and Jarník (1931) theorem.

1.2 Improvements to asymptotic Diophantine approximation result

In this section we give details about theorems by Hurwitz (1891), Khintchine (1924), Jarník (1931) and Jarník–Besicovitch (1928, 1934).

The starting point for improving (1.9) is Hurwitz's Theorem (1891). Hurwitz sharpened (1.9) by showing that the rate $\frac{1}{q}$ can be replaced by $\frac{1}{\sqrt{5}q}$.

Theorem 1.2.1 (Hurwitz, [26]) *For any irrational $x \in \mathbb{R}$, there exist infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ satisfying*

$$|qx - p| < \frac{1}{\sqrt{5}q}.$$

Moreover, the constant $\frac{1}{\sqrt{5}}$ is the best possible. ■

The latter part just means that Hurwitz's statement becomes false if $\frac{1}{\sqrt{5}}$ is replaced by any smaller constant say $\frac{1}{\sqrt{5+\delta}}$, for any arbitrary $\delta > 0$. In particular for the Golden ratio $x_1 = \frac{1+\sqrt{5}}{2}$, the inequality

$$|qx_1 - p| < \frac{1}{(\sqrt{5}+\delta)q}$$

holds for only finitely many $(p,q) \in \mathbb{Z} \times \mathbb{N}$.

If from the set of irrationals we exclude the Golden ratio then $\frac{1}{\sqrt{5}}$ can be improved by $\frac{1}{2\sqrt{2}}$ which is the best possible for $x_2 = 1+\sqrt{2}$. Similarly, by ignoring the irrational $x_2 = 1+\sqrt{2}$. The constant $\frac{1}{2\sqrt{2}}$ can be reduced further to $\frac{5}{\sqrt{221}}$ which is optimal for $x_3 = \frac{9+\sqrt{221}}{10}$. The story does not stop here, for a more extended list see [10]. Continuing in this way, we find successively smaller constants and the collection of all such best constants is known as the Markoff spectrum. It is proved in [12] that the sequence of associated best constants tends to $\frac{1}{3}$ and cannot be reduced further via the same method. To establish what happens when if we consider constant to be smaller than $\frac{1}{3}$ we require some more sophisticated theorems. This discussion shows that there are real numbers x for which (1.9) cannot be improved beyond a constant, thus leading us to the theory of badly approximable numbers.

A real number x is said to be badly approximable if there exist a constant $\mathcal{C} > 0$ depending on x, such that

$$|qx - p| > \frac{\mathcal{C}}{q},$$

for all $(p,q) \in \mathbb{Z} \times \mathbb{N}$. Denote the set of badly approximable numbers by **Bad**. Clearly, **Bad** $\neq \emptyset$, since $\frac{1+\sqrt{5}}{2} \in$ **Bad**.

Example 1.2.2 $\sqrt{2}$ is badly approximable. ■

Proof: Consider the polynomial,

$$\mathcal{P}(\alpha) := \alpha^2 - 2 = (\alpha - \sqrt{2})(\alpha + \sqrt{2}) = (\sqrt{2} - \alpha)(-\sqrt{2} - \alpha).$$

Assume that, for some positive constant $\mathcal{C}$,

$$\left|\sqrt{2} - \frac{p}{q}\right| < \frac{\mathcal{C}}{q^2}, \tag{1.10}$$

holds for infinitely many $(p,q) \in \mathbb{Z} \times \mathbb{N}$. Then

$$\begin{aligned}\left|\mathcal{P}\left(\frac{p}{q}\right)\right| &= \left|\sqrt{2} - \frac{p}{q}\right|\left|\sqrt{2} + \frac{p}{q}\right| = \left|\sqrt{2} - \frac{p}{q}\right|\left|\sqrt{2} + \frac{p}{q} - \sqrt{2} + \sqrt{2}\right| \\ &\leq \left|\sqrt{2} - \frac{p}{q}\right|\left(2\sqrt{2} + \left|\sqrt{2} - \frac{p}{q}\right|\right) < \frac{2\sqrt{2}\mathcal{C}}{q^2} + \frac{\mathcal{C}^2}{q^4}.\end{aligned}$$

Since the polynomial $\mathcal{P}$ is irreducible, we get on the other side

$$\left|\mathcal{P}\left(\frac{p}{q}\right)\right| = \frac{|p^2 - 2q^2|}{q^2} \geq \frac{1}{q^2}.$$

Comparing both estimates for $\mathcal{P}\left(\frac{p}{q}\right)$, we obtain

$$1 < 2\sqrt{2}\mathcal{C} + \frac{\mathcal{C}^2}{q^2}.$$

This implies $\mathcal{C} \geq \frac{1}{2\sqrt{2}}$ by letting $q \to \infty$. Thus for $\mathcal{C} \geq \frac{1}{2\sqrt{2}}$, the inequality (1.10) has infinitely many solutions and for $\mathcal{C} < \frac{1}{2\sqrt{2}}$ the inequality (1.10) has finitely many solutions. ■

In fact all *quadratic irrationals* are in **Bad**, (see Corollary 1.2) which is the consequence of a beautiful characterisation that the set **Bad** can be completely determined in terms of the theory of continued fractions.

According to a classical result a real number x is badly approximable if and only if the partial quotients in its continued fraction expansion are bounded. More precisely if $x = [a_0; a_1, a_2, a_3, \ldots]$ is the continued fraction representation of x then

$$x \in \textbf{ Bad} \iff \exists\, K(x) > 0 \ :\ |a_n| \leq K(x) \quad \text{for all } n \ \geq 1. \tag{1.11}$$

A real number x is quadratic irrational if and only if it has periodic continued fraction representation that is

$$x = [a_0; a_1, a_2, a_3, \ldots, a_i, \overline{a_{i+1}, \ldots, a_{i+n}}].$$

Moreover, if a real number is periodic then the partial quotients in its continued fraction expansion are bounded by a constant. All these arguments along with (1.11) lead to the following result.

Corollary 1.2.3 *All quadratic irrationals are badly approximable.*

An obvious example is $\frac{1+\sqrt{5}}{2}$ i.e., the Golden ratio, which is a root of the quadratic equation $x^2 - x - 1$ and has the simplest continued fraction representation

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \cdots],$$

which is periodic and thus bounded. In fact, we have a more general result as follows.

Theorem 1.2.4 *Suppose $x \in \mathbb{R}$ is the root of an irreducible nonzero quadratic polynomial $c_1X^2 + c_2X + c_3$ where $c_1, c_2, c_3 \in \mathbb{Z}$. Then for any $\mathcal{C} < \frac{1}{\sqrt{c_2^2 - 4c_1c_3}}$ the inequality*

$$|qx - p| < \frac{\mathcal{C}}{q},$$

has only finitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

In contrast with the idea of badly approximable numbers we can consider irrationals which are well approximated by rationals. That is, if we examine the exponent on the denominator 'q' in the right hand side of (1.9) then exponent '1' is best possible in the sense that if it is replaced by any exponent $\tau > 1$ then (1.9) will not be true for all irrationals. To elaborate this situation consider a real number $\tau > 0$ and let $\mathcal{W}(\tau)$ be the set of real numbers x such that there are infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ satisfying

$$|qx - p| < \frac{1}{q^\tau}.$$

The members of the set $\mathcal{W}(\tau)$ are known as *τ-well approximable numbers*. Thus, the set of τ-well approximable numbers consist of the real numbers for which the exponent $\tau > 1$ in (1.9) can be improved. From Dirichlet's Theorem trivially we have $\mathcal{W}(\tau) = \mathbb{R}$ for $\tau \leq 1$.

Clearly, the inclusion $\mathcal{W}(\tau_1) \subset \mathcal{W}(\tau)$ is obvious for any $\tau < \tau_1$. Further the quality of approximation may be measured in terms of the so called irrationality exponent which for any irrational x, is defined by

$$\tau(x) = \sup\{\tau \in \mathbb{R} \ : x \in \mathcal{W}(\tau)\}.$$

As a result of Dirichlet's Theorem, $\tau(x)$ is always bounded below by one. The numbers for which $\tau(x)$ is larger than one are known as τ-well approximable numbers.

Then there are Liouville numbers. Such numbers are very well approximable which means they are contained in $\mathcal{W}(\tau)$ for arbitrary large τ and the collection of such numbers is denoted by $\mathcal{L} = \bigcap_{\tau > 0} \mathcal{W}(\tau)$. The set $\mathcal{L}$ is non-empty since

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000000100\ldots \quad \in \mathcal{W}(\tau).$$

The sets **Bad** and $\mathcal{W}(\tau)$ provide good points of reference as they represent two extremes of approximation. So the natural question is how large are the sets **Bad** and $\mathcal{W}(\tau)$. Equivalently, we ask how likely it is that a given real number is contained in one of these sets. In simple words we are interested in the 'size' of these sets.

Metric Diophantine approximation: Lebesgue measure

One of the primary ideas of 'size' which can be considered to measure sets like $\mathcal{W}(\tau)$ is Lebesgue measure. Loosely speaking, we say that a set of real numbers has 'full' Lebesgue measure if a randomly chosen real number lies in the set with probability one. Throughout the thesis, for any measurable set $U \subset \mathbb{R}$, we will denote its Lebesgue measure by $\lambda(U)$.

Next we generalise the idea of τ-well approximable numbers. For this consider a decreasing function $\psi : \mathbb{N} \to \mathbb{R}^+$, such that $\psi(q) \to 0$ as $q \to \infty$ and call it an *approximating function.* The approximating function ψ controls the rate at which irrationals are approximated by rationals.

Denote by $\mathcal{W}(\psi)$ the set of all real numbers x which satisfy the inequality

$$|qx - p| < \psi(q)$$

for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$. The members of $\mathcal{W}(\psi)$ are usually referred to as *ψ-well approximable numbers.*

Obviously, when $\psi : q \mapsto \psi(q) := q^{-\tau}$ the set $\mathcal{W}(\psi)$ is simply the set $\mathcal{W}(\tau)$ of τ-well approximable numbers.

Further, note that as a consequence of Corollary 1.1.3,

$$\mathcal{W}(\psi) = \mathbb{R} \quad \text{if} \quad \psi(q) = \frac{1}{q}.$$

The structure of sets like $\mathcal{W}(\psi)$ and **Bad** look almost similar to fractals. Therefore the foremost problem of the metrical theory of Diophantine approximation was to explore the sizes of sets **Bad** and $\mathcal{W}(\psi)$ in terms of Lebesgue measure and later in terms of Hausdorff measure and dimension. Without any loss of generality, we will focus on real numbers in the unit interval $[0, 1)$ unless mentioned otherwise.

Note that the set $\mathcal{W}(\psi)$ can be viewed as a limsup set of balls. Let $q \in \mathbb{N}$ be fixed, then

$$\mathcal{W}(\psi) = \limsup_{q\to\infty} U_q = \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} U_q(\psi) \tag{1.12}$$

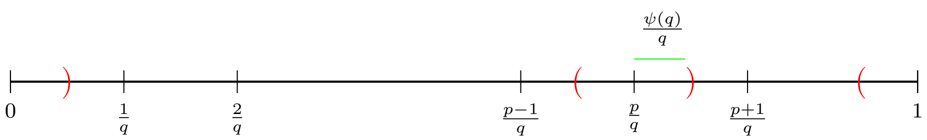
where

$$\begin{aligned} U_q(\psi) &:= \bigcup_{p=0}^{q} \left\{ x \in [0,1) : |qx - p| < \psi(q) \right\} \\ &:= \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1) \\ &= \bigcup_{p=0}^{q} \left[\frac{p-\psi(q)}{q}, \frac{p+\psi(q)}{q}\right] \cap [0,1) \end{aligned}$$

Also observe that,

$$\lambda(U_q(\psi)) = 2\frac{\psi(q)}{q} \leq 2\psi(q), \tag{1.13}$$

signifying the disjointness of the intervals in $U_q(\psi)$ for $\psi(q) < \frac{1}{2}$.



Clearly from the figure we have $q-1$ full intervals of length $\frac{2\psi(q)}{q}$ whereas on the end of the line we have two half intervals. From (1.12) for any $N \in \mathbb{N}$,

$$\mathcal{W}(\psi) \subseteq \bigcup_{q=N}^{\infty} U_q(\psi).$$

Thus (1.13) implies

$$\lambda(\mathcal{W}(\psi)) \leq 2\sum_{q=N}^{\infty} \psi(q). \tag{1.14}$$

Next assume

$$\sum_{q=N}^{\infty} \psi(q) < \infty.$$

Then for any $\delta > 0$, there exists N_0 such that for all $N \geq N_0$, we have

$$\sum_{q=N}^{\infty} \psi(q) < \frac{\delta}{2}.$$

Then from (1.14),

$$\lambda(\mathcal{W}(\psi)) < \delta,$$

and from the arbitrariness of δ it follows that

$$\lambda(\mathcal{W}(\psi)) = 0.$$

Thus we have the following corollary.

Corollary 1.2.5 *Let ψ be same as defined above and $\sum_{q=1}^{\infty} q\psi(q) < \infty$. Then $\lambda(\mathcal{W}(\psi)) = 0$.*

The above statement which is generally known as the convergence case for Khintchine's Theorem is a consequence of the Borel–Cantelli Lemma from probability theory.

Lemma 1.2.6 (Borel–Cantelli) Let $(\mathcal{X}, \Sigma, \mu)$ be a measure space with probability measure $\mu(\mathcal{X}) < \infty$, and let $\{U_q\}_{q\geq 1} \subseteq \Sigma$ be a sequence of events in Σ. If

$$\sum_{q=1}^{\infty} \mu(U_q) < \infty, \tag{1.15}$$

then

$$\mu\left(\limsup_{q\to\infty} U_q\right) = 0.$$

Proof: For any $N \geq 1$, $\limsup_{q\to\infty} U_q$ is contained in $\bigcup_{q=N}^{\infty} U_q$. Therefore,

$$\mu\left(\limsup_{q\to\infty} U_q\right) = \mu\Big(\bigcap_{N=1}^{\infty}\bigcup_{q=N}^{\infty} U_q\Big) \leq \mu\Big(\bigcup_{q=N}^{\infty} U_q\Big) \leq \sum_{q=N}^{\infty} \mu(U_q). \tag{1.16}$$

In view of (1.15), the last inequality in (1.16) tends to 0 as $N \to \infty$. ■

Next if we consider the situation where $\sum_{q=1}^{\infty}\mu(U_q)$ diverges. Then does this implies $\mu(\limsup_{q\to\infty} U_q) > 0$ or $\mu(\limsup_{q\to\infty} U_q) = 0$? Unfortunately, in general it is not true as there is a possibility of constructing nested intervals $\{U_q\}_{q\geq 1}$ such that the sum diverges but the measure is zero as demonstrated by the following example.

Example 1.2.7 Consider $U_q := \left(0, \frac{1}{q}\right)$ and assume probability measure 'μ' to be Lebesgue measure. Then

$$\sum_{q=1}^{\infty}\lambda(U_q) = \sum_{q=1}^{\infty}\frac{1}{q} = \infty.$$

Whereas, for any $N \in \mathbb{N}$ we have

$$\limsup_{q\to\infty} U_q = \bigcap_{N=1}^{\infty}\bigcup_{q=N}^{\infty}\left(0, \frac{1}{q}\right) = \bigcap_{N=1}^{\infty}\left(0, \frac{1}{N}\right) = \emptyset.$$

Therefore,

$$\lambda(\limsup_{q\to\infty} U_q) = 0.$$

Recall that Corollary 1.2.5 implies that if ψ decreases rapidly enough such that the 'measure' sum converges then the set of ψ-well approximable numbers is an exceptional set (i.e., of Lebesgue measure zero). Next think of the case when the measure sum diverges, then what will be the 'size' of the set $\mathcal{W}(\psi)$?

The following groundbreaking measure-theoretic result by Khintchine in 1924 sits at the heart of metric Diophantine approximation and is known as Khintchine's Theorem. It gives an elegant 'zero-full' law for the 'size' of $\mathcal{W}(\psi)$ expressed in terms of one-dimensional Lebesgue measure.

Theorem 1.2.8 (Khintchine, [30]) *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function with $\psi(q) \to 0$ as $q \to \infty$. Then*

$$\lambda(\mathcal{W}(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty}\psi(q) < \infty, \\ \\ 1 & \text{if} \quad \sum_{q=1}^{\infty}\psi(q) = \infty. \end{cases}$$

The divergence case plays the foremost role in Khintchine's Theorem and that is where the decreasing (monotonic) condition for ψ is compulsory. It is noteworthy that in the original statement Khintchine imposed the stricter assumption that $q\psi(q)$ is decreasing. The fact that this extra assumption is not required is because of [5]. However the condition that ψ is decreasing is considered essential for the divergent case. This point was highlighted by Duffin–Schaeffer in [18]. To support this argument they established a counterexample which involved a non-monotonic function $\upsilon : \mathbb{N} \to \mathbb{R}^+$ such that $\lambda(\mathcal{W}(\upsilon)) = 0$ but $\sum_{q=1}^{\infty}\upsilon(q) = \infty$.

The counterexample produced by Duffin–Schaeffer was built on the fact that the following product taken over all primes p diverges, i.e.,

$$\prod_p \Big(1+\frac{1}{p}\Big) = \infty.$$

Thus there is a sequence $\{z_i\}_{i\geq 1}$ of positive square free integers such that $(z_i, z_k) = 1$ for $i \neq k$, and

$$\prod_{p|z_i} \Big(1+\frac{1}{p}\Big) > 2^i + 1.$$

Let

$$\upsilon(q) := \begin{cases} \frac{2^{-i-1}q}{z_i} & \text{if} \quad q \mid z_i \text{ for some } i \text{ and } q > 1 \\ \\ 0 & \text{if} \quad q \nmid z_i, \text{ or } \quad q = 1, \end{cases}$$

and define U_q by

$$U_q := U_q(\upsilon) = \bigcup_{p=0}^{q} \Big[\frac{p-\upsilon(q)}{q}, \frac{p+\upsilon(q)}{q}\Big] \cap [0,1).$$

For $q > 1$, if $q \mid z_i$ then $U_q \subseteq U_{z_i}$ and so

$$\bigcup_{q|z_i} U_q = U_{z_i}.$$

Also for this case

$$\lambda(U_q) = 2\frac{\upsilon(q)}{q} = \frac{2^{-i}}{z_i}.$$

Therefore,

$$\lambda\Big(\bigcup_{q|z_i} U_q\Big) = \lambda(U_{z_i}) = 2\upsilon(z_i) = 2^{-i} \quad \text{for} \quad i = 1, 2, \ldots .$$

Consider an arbitrary $n \in \mathbb{N}$ such that

$$\mathcal{W}(\upsilon) \subseteq \bigcup_{i=n}^{\infty} \bigcup_{q|z_i} U_q,$$

then

$$\lambda(\mathcal{W}(\upsilon)) \leq \sum_{i=n}^{\infty} 2^{-i} = 2^{-n+1}.$$

Consequently,

$$\lambda(\mathcal{W}(\upsilon)) = 0.$$

Next observe that,

$$\begin{aligned}\sum_{q=1}^{\infty} \upsilon(q) &= \sum_{i=1}^{\infty} 2^{-i-1}\frac{1}{z_i}\sum_{q>1,\ q|z_i} q\\ &= \sum_{i=1}^{\infty} 2^{-i-1}\frac{1}{z_i}\prod_{p|z_i}(1+p)\\ &= \sum_{i=1}^{\infty} 2^{-i-1}\Big(\prod_{p|z_i}(1+\frac{1}{p})\Big)\\ &\geq \sum_{i=1}^{\infty} 2^{-i-1}(2^i+1) = \infty.\end{aligned}$$

However, in the same paper [18], Duffin–Schaffer proposed a more suitable statement for any arbitrary function ψ. They started by defining the set of real number x such that

$$|qx - p| < \psi(q), \tag{1.17}$$

for infinitely many $(p,q) \in \mathbb{Z}\times\mathbb{N}$ with $(p,q) = 1$. Let $\mathcal{W}'(\psi)$ represent the collection of all such points x. Then clearly, $\mathcal{W}'(\psi) \subset \mathcal{W}(\psi)$. Therefore $\lambda(W(\psi)) = 0$ implies $\lambda(W'(\psi)) = 0$ so the convergence case for Khintchine's Theorem stays valid for $\mathcal{W}'(\psi)$ replacing $\mathcal{W}(\psi)$. Further it is straightforward to deduce that

$$\lambda(\mathcal{W}'(\psi)) = 0 \text{ if } \quad \sum_{q=1}^{\infty}\Theta(q)\frac{\psi(q)}{q} < \infty,$$

where Θ is the *Euler* function, (in number theory, Euler's function counts the positive integers up to a given integer n that are relatively prime to n). The co-primality condition imposed on p and q in (1.17) guarantees that the rational $\frac{p}{q}$ to x with rate of approximation $\frac{\psi(q)}{q}$ are in reduced form. Then the conjecture is as follows.

Conjecture 1.2.9 (Duffin–Schaeffer, [18]) For any function $\psi : \mathbb{N} \to \mathbb{R}^+$,

$$\lambda(\mathcal{W}'(\psi)) = 1 \text{ if } \quad \sum_{q=1}^{\infty}\Theta(q)\frac{\psi(q)}{q} = \infty.$$

The profound Duffin–Schaeffer Conjecture 1941, which remained open until solved by Koukoulopoulos–Maynard [37] in 2019.

Theorem 1.2.10 (Koukoulopoulos–Maynard, [37]) *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be any function. Then*

$$\lambda(\mathcal{W}'(\psi)) = 1 \textit{ if } \quad \sum_{q=1}^{\infty}\Theta(q)\frac{\psi(q)}{q} = \infty.$$

Returning briefly to the set **Bad**. From Khintchine's Theorem and by a simple observation

$$\mathbf{Bad} \subseteq [0,1)\backslash\mathcal{W}\left(\psi : q \mapsto \frac{1}{q\log q}\right).$$

Thus it follows that

$$\lambda(\mathbf{Bad}) = 0.$$

Similarly, the set of very well approximable numbers has Lebesgue measure zero.

If we jump back to the set $\mathcal{W}(\tau)$ of τ-well approximable numbers then recall that the convergence case of Khintchine's Theorem implies that $\lambda(\mathcal{W}(\tau)) = 0$ for any $\tau > 1$. Heuristically, one would expect that as τ increases the 'size' of $\mathcal{W}(\tau)$ will decrease. Particularly, it is expected that the set say $\mathcal{W}(200)$ is smaller than the set $\mathcal{W}(20)$, since $\mathcal{W}(200) \subset \mathcal{W}(20)$. But both of them are exceptional sets. Thus Lebesgue measure fails to distinguish between them from a metric point of view. Therefore we might be interested in more elegant notion of measuring size. One such method is Hausdorff measure and dimension.

Metric Diophantine Approximation: Hausdorff measure and dimension

Hausdorff dimension is the refinement of our intuitive idea of dimension but it does not necessarily takes integer value and therefore gives us a method for allocating reasonable values of dimension to, say, fractal sets. It depends on covering a set by small sets.
To define the Hausdorff dimension of the set we need to know the concept of Hausdorff measure. Hausdorff measure generalises the familiar notions of length, area, volume, etc. It might be demonstrated that for a subsets of $\mathbb{R}^n$, the n-dimensional Hausdorff measure is, to within a constant multiple, just an n-dimensional Lebesgue measure. Thus Hausdorff measure is a generalisation of Lebesgue measure and Hausdorff dimension is a generalisation of Euclidean (integer) dimension.

In what follows, a *dimension function* is an increasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \to 0$ as $r \to 0$ and $\mathcal{V}$ is an arbitrary subset of $\mathbb{R}^n$. For $\rho > 0$, a ρ-cover for a set $\mathcal{V}$ is defined as the countable collection $\{U_i\}_{i\geq 1}$ of sets in $\mathbb{R}^n$ with diameters $0 < \operatorname{diam}(U_i) \leq \rho$ such that $\mathcal{V} \subset \bigcup_{i=1}^{\infty} U_i$ for each i. Then for each $\rho > 0$ define

$$\mathcal{H}^f_\rho(\mathcal{V}) = \inf\left\{\sum_{i=1}^{\infty} f\big(\operatorname{diam}(U_i)\big) : \{U_i\} \text{ is a } \rho\text{ -cover of } \mathcal{V}\right\}.$$

Note that $\mathcal{H}^f_\rho(\mathcal{V})$ increases as ρ decreases and therefore approaches a limit. Accordingly, the f-dimensional Hausdorff measure of $\mathcal{V}$ is defined as

$$\mathcal{H}^f(\mathcal{V}) := \lim_{\rho\to 0} \mathcal{H}^f_\rho(\mathcal{V}).$$

This limit could be zero or infinity, or take a finite positive value.

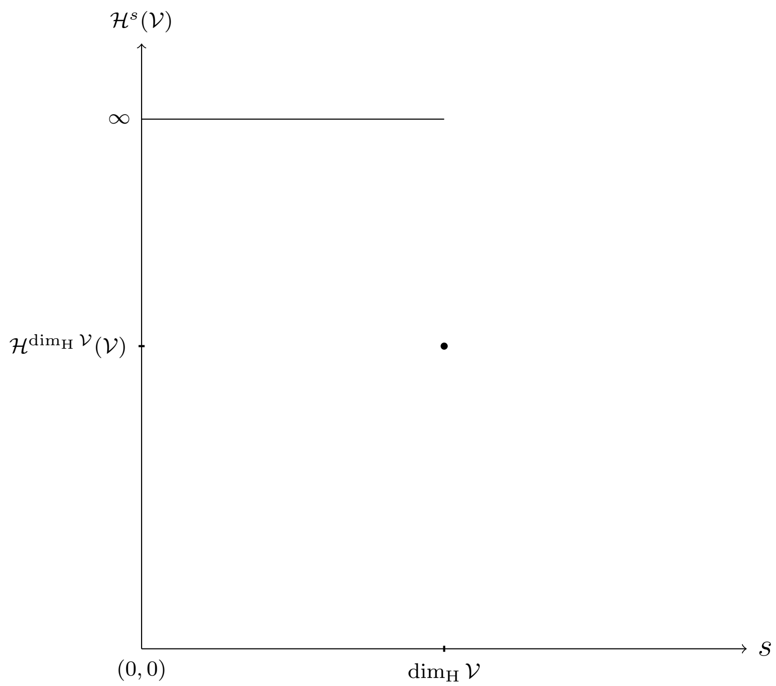
If $f(r) = r^s$ where $s \geq 0$, then $\mathcal{H}^f$ is the s-dimensional Hausdorff measure and is represented by $\mathcal{H}^s$. It can be easily verified that Hausdorff measure is monotonic, countably sub-additive, and $\mathcal{H}^s(\emptyset) = 0$.

The following property

$$\mathcal{H}^s(\mathcal{V}) < \infty \implies \mathcal{H}^{s'}(\mathcal{V}) = 0 \quad \text{if } s' > s,$$

implies that there is a unique real point s at which the Hasudorff s-measure drops from infinity to zero (unless $\mathcal{V}$ is finite so that $\mathcal{H}^s(\mathcal{V})$ is never finite). The value taken by s at this discontinuity is referred to as the *Hausdorff dimension* of a set $\mathcal{V}$ and is defined as

$$\dim_{\mathrm{H}} \mathcal{V} := \inf\{s \geq 0 : \ \mathcal{H}^s(\mathcal{V}) = 0\}.$$



Graph of Hausdorff measure $\mathcal{H}^s(\mathcal{V})$ against the exponent s.

From the definition of $\dim_{\mathrm{H}} \mathcal{V}$, we have

$$\mathcal{H}^s(\mathcal{V}) = \begin{cases} 0 & \text{if } s > \dim_{\mathrm{H}} \mathcal{V}, \\ \infty & \text{if } s < \dim_{\mathrm{H}} \mathcal{V}. \end{cases}$$

If $s = \dim_{\mathrm{H}} \mathcal{V}$, then $\mathcal{H}^s$ may be 0 or ∞ or may take a finite value i.e., $\mathcal{H}^s \in (0,\infty)$. When $s = n$, $\mathcal{H}^n$ coincides with standard Lebesgue measure on $\mathbb{R}^n$.

Computing the Hausdorff dimension of a set (say $\dim_{\mathrm{H}} \mathcal{V} = s$) is typically accomplished in two steps: obtaining the upper bounds i.e., showing $\dim_{\mathrm{H}} \mathcal{V} \leq s$ and the lower bounds i.e., $\dim_{\mathrm{H}} \mathcal{V} \geq s$, separately. In most cases upper bounds are easier to obtain since it is enough to provide specific covers as $\rho \to 0$. Usually lower bounds are harder to establish as one needs to work with all possible ρ-covers of $\mathcal{V}$ in order to get the infimum value.

The Jarník–Besicovitch Theorem

Recall that $\mathcal{W}(\psi)$ for the approximating function $\psi(q) = q^{-\tau}$ is the set $\mathcal{W}(\tau)$. From the convergence case of Theorem 1.2.8, for any $\tau > 1$ we have $\lambda(\mathcal{W}(\tau)) = 0$. One would expect that as we increase τ the 'size' of $\mathcal{W}(\tau)$ decreases. The following result by Jarník [29] and then independently proved by Besicovitch [8], known as Jarník–Besicovitch Theorem tells that for any $\tau \geq 1$,

$$\dim_{\mathrm{H}} \mathcal{W}(\tau) = \frac{2}{1+\tau}.$$

From this result it is obvious that

$$\dim_{\mathrm{H}} \mathcal{W}(20) = \frac{2}{21} \quad \text{and} \quad \dim_{\mathrm{H}} \mathcal{W}(200) = \frac{2}{201}$$

and so $\mathcal{W}(200)$ is 'smaller' than $\mathcal{W}(20)$ as expected. Thus for sets like $\mathcal{W}(\tau)$ Hausdorff dimension is a source to distinguish between their sizes. Lebesgue measure fails to do.

Jarník–Besicovitch Theorem is a fantastic dimension result however it does not gives any additional informations regards to $\mathcal{H}^s$ at the critical value $\wp := \frac{2}{1+\tau}$. By definition

$$\mathcal{H}^s(\mathcal{W}(\tau)) = \begin{cases} 0 & \text{if} \quad s > \wp, \\ \infty & \text{if} \quad s < \wp, \end{cases}$$

but

$$\mathcal{H}^s(\mathcal{W}(\tau)) = ? \quad \text{if } s = \wp.$$

That is, it fails to differentiate between sets having same dimension. Take, for example, the approximating functions

$$\psi_1(q) = \frac{1}{q^{10}} \quad \text{and} \quad \psi_2(q) = \frac{1}{q^{10}\log q}.$$

Then from the Jarník–Besicovitch Theorem we have

$$\dim_{\mathrm{H}} \mathcal{W}(\psi_1) = \frac{2}{11} \quad \text{and} \quad \dim_{\mathrm{H}} \mathcal{W}(\psi_2) = \frac{2}{11}.$$

Briefly we are interested to seek the Hausdorff measure analogue of Theorem 1.2.8.

Jarník's Theorem

The following elegant statement by Jarník is concerned with the Hausdorff measure of $\mathcal{W}(\psi)$.

Theorem 1.2.11 (Jarník, [29]) *Let ψ be an approximating function and $s \in (0,1)$. Then*

$$\mathcal{H}^s(\mathcal{W}(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q\left(\frac{\psi(q)}{q}\right)^s < \infty, \\ \infty & \text{if} \quad \sum_{q=1}^{\infty} q\left(\frac{\psi(q)}{q}\right)^s = \infty. \end{cases}$$

With $\psi(q) = q^{-\tau}$ $(\tau > 1)$, Theorem 1.2.11 not only tells that $\dim_{\mathrm{H}} \mathcal{W}(\tau) = \frac{2}{1+\tau}$ but also reveals the critical exponent at which the Hausdorff measure is infinite, i.e.

$$\mathcal{H}^s(\mathcal{W}(\tau)) = \infty \text{ at } s = \frac{2}{1+\tau}.$$

The case $s = 1$ can be naturally excluded since

$$\mathcal{H}^1(\mathcal{W}(\psi)) \asymp \lambda(\mathcal{W}(\psi)) = 1.$$

1.3 Improvements to uniform Diophantine approximation result

The previous discussion demonstrates that generalisations of Corollary 1.1.3 have been well studied over the years. As discussed earlier Khintchine's Theorem gives precise conditions for the Lebesgue measure of $\mathcal{W}(\tau)$. Quite surprisingly no such clean statement has been proved (until 2018) in the set-up of Theorem 1.1.2. In connection with Theorem 1.1.2 we have the set of Dirichlet improvable numbers.

From now onwards let $\psi : [t_0, \infty) \to \mathbb{R}^+$ be a decreasing function with $t_0 \geq 1$ fixed and let $t\psi(t) < 1$ for all $t \geq t_0$. Define the set $\mathcal{D}(\psi)$ of ψ-Dirichlet improvable numbers by

$$\mathcal{D}(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\, N \text{ such that the system } |qx - p| < \psi(t), |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\}. \tag{1.18}$$

Note that this definition emerges from (1.7) by replacing '$\leq \frac{1}{t}$' with '$< \psi(t)$' and aiming for the existence of nontrivial integer solution for all large t except those belonging to a bounded set. The members of the complementary set $\mathcal{D}(\psi)^c$ are known as ψ-Dirichlet non-improvable numbers. Since Theorem 1.1.2 was proved by a simple pigeon-hole argument, there should be a large room for improvement. So the natural question is what is the metrical theory associated with the set $\mathcal{D}(\psi)^c$?

Chapter 2

Auxiliary results

In this chapter we mention some elementary results and various techniques which will be helpful in establishing the main results of this thesis. Just to avoid confusion, throughout the thesis we can also use a_n, p_n and q_n in place of $a_n(x)$, $p_n(x)$ and $q_n(x)$, respectively.

2.1 Continued fractions and Diophantine approximation

From Chapter 1, it is obvious that continued fractions play a very important role in the metrical theory of Diophantine approximation. In this section we will recall some basic properties of continued fractions in connection with Diophantine approximation. They are explained in the standard texts [28, 31].

For any integer vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $n \geq 1$, define

$$I_n(a_1, \ldots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}$$

and call it the *basic cylinder* of order n. Note that I_n simply represents the set of all real numbers in $[0, 1)$ whose continued fraction expansions begin with $(a_1, \ldots, a_n)$.

Proposition 2.1.1 *For any positive integers $a_1, \ldots, a_n$, let $p_n = p_n(a_1, \ldots, a_n)$ and $q_n = q_n(a_1, \ldots, a_n)$ be defined recursively by (1.2). Then:*

1.

$$I_n(a_1, a_2, \ldots, a_n) = \begin{cases} \left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if} \quad n \text{ is even;} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right] & \text{if} \quad n \text{ is odd ,} \end{cases} \tag{2.1}$$

and by using (1.3) we have

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}, \tag{2.2}$$

where throughout the thesis $|\star|$ *represents the length of any interval.*

2. *For any* $n \geq 1$*, we have*
$$q_n \geq 2^{(n-1)/2} \tag{2.3}$$
and for any $1 \leq k \leq n$
$$\frac{a_k+1}{2} \leq \frac{q_n(a_1,\dots,a_n)}{q_n(a_1,\dots,a_{k-1},a_{k+1}\dots,a_n)} \leq a_k+1.$$

3. *For any* $n \geq 1$
$$\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, \dots, a_1].$$
and $k \geq 1$*, we have*
$$q_{n+k}(a_1,\dots,a_n,a_{n+1}\dots,a_{n+k}) \geq q_n(a_1,\dots,a_n)q_k(a_{n+1},\dots,a_{n+k}), \tag{2.4}$$
$$q_{n+k}(a_1,\dots,a_n,a_{n+1}\dots,a_{n+k}) \leq 2q_n(a_1,\dots,a_n)q_k(a_{n+1},\dots,a_{n+k}). \tag{2.5}$$

4. *A simple fact on continued fraction gives that*
$$\left|q_{n-1}(x)x - p_{n-1}(x)\right| = \frac{1}{q_n(x)+T^n(x)\cdot q_{n-1}(x)} = \frac{1}{q_n(x)(1+T^n(x)\cdot\frac{q_{n-1}(x)}{q_n(x)})},$$

5. *Speed of approximation is given by the following formula,*
$$\frac{1}{3a_{n+1}(x)q_n^2(x)} < \left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{1}{q_n(x)(q_{n+1}(x)+T^{n+1}(x)q_n(x))} < \frac{1}{a_{n+1}(x)q_n^2(x)}, \tag{2.6}$$

and the derivative of T^n *is given by*
$$(T^n)'(x) = \frac{(-1)^n}{(xq_{n-1}(x)-p_{n-1}(x))^2}.$$
Further,
$$q_n^2(x) \leq \prod_{k=0}^{n-1}|T'(T^k(x))| \leq 4q_n^2(x). \tag{2.7}$$

From (1.2) note that q_n is determined by $a_1,\dots,a_n$ for any $n \geq 1$. Therefore, we can write $q_n = q_n(a_1,\dots,a_n)$.

The next proposition describe the positions of cylinders I_{n+1} of order $n+1$ inside the nth order cylinder I_n.

Proposition 2.1.2 **([31])** *Let* $I_n = I_n(a_1,\dots,a_n)$ *be a basic cylinder of order* n*, which is partitioned into sub-cylinders* $\{I_{n+1}(a_1,\dots,a_n,a_{n+1}) : a_{n+1} \in \mathbb{N}\}$*. When* n *is odd, these sub-cylinders are positioned from left to right, as* a_{n+1} *increases from* 1 *to* ∞*; when* n *is even, they are positioned from right to left.*

As discussed in Chapter 1, according to Legendre's Theorem if an irrational x is well approximated by a rational $\frac{p}{q}$, then this rational must be a convergent of x. Thus in order to find good rational approximates to an irrational number we only need to focus on its convergents. Note that from (2.6), a real number x is well approximated by its convergent $\frac{p_n}{q_n}$ if its $(n+1)$th partial quotient is sufficiently large.

The next result is due to Łuczak [39].

Lemma 2.1.3 (Łuczak, [39]) For any $a, b > 1$, the sets

$$\left\{x \in [0,1) : a_n(x) \geq a^{b^n} \text{ for infinitely many } n \in \mathbb{N}\right\}$$

and

$$\left\{x \in [0,1) : a_n(x) \geq a^{b^n} \text{ for all sufficiently large } n \in \mathbb{N}\right\}$$

are of the same Hausdorff dimension $\frac{1}{1+b}$. ■

The mass distribution principle

As discussed earlier deriving the Hausdorff dimension for any set, normally consists of two parts: obtaining the upper and the lower bounds separately. The upper bound usually follows by using a suitable covering argument whereas estimation of lower bounds needs clever synthesis of the set supporting a certain outer measure on the set under study. The next simple but crucial result, which will be the main ingredient for finding the lower bounds of sets in this thesis, is commonly known as the mass distribution principle [19, §4.2].

Proposition 2.1.4 (Mass Distribution Principle) *Let $\mathcal{U} \subset [0,1)$ have a positive measure $\mu(\mathcal{U}) > 0$ and suppose that for some $s > 0$ there exists a constant $c > 0$ such that if for any $x \in [0,1)$, we have*

$$\mu(B(x,r)) \leq cr^s,$$

where $B(x,r)$ denotes an open ball centred at x and radius r. Then $\dim_{\mathrm{H}} \mathcal{U} \geq s$. ■

2.2 Pressure function and Hausdorff dimension

In this section we recall the definition of a *pressure function* and go through some of its basic properties. It plays an important role for finding Hausdorff dimension of sets connected with the properties of continued fractions.

The general idea of pressure function, specially the topological pressure, is comprehensively explained in Walters' [44] book. For the requirement of this thesis we just need to focus on that area which specialises in the settings of continued fraction. Directing the reader through the references, the end game is to produce a function, from which we can produce a lower bound for the Hausdorff dimension of our sets of interest.

The pressure function naturally gets involved in the dimension theory for conformal iterated function systems (for examples, see [40]).

Mauldin–Urbański [41] presented a form of pressure function in conformal iterated function systems with applications to the geometry of continued fractions, for more thorough details we refer to [40, 41, 42].

Consider a finite or infinite subset $\mathcal{A}$ of natural numbers and define

$$\mathcal{Z}_{\mathcal{A}} = \{x \in [0,1) : \text{ for all } n \geq 1, a_n(x) \in \mathcal{A}\}.$$

Then $(\mathcal{Z}_{\mathcal{A}}, T)$ is a subsystem of $([0,1), T)$ where T is a Gauss map as defined in (1.1). Let $\varphi : [0,1) \to \mathbb{R}$ be a real function. Then the pressure function with respect to potential φ and restricted to the system $(\mathcal{Z}_{\mathcal{A}}, T)$ is defined as

$$\mathsf{P}_{\mathcal{A}}(T, \varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\cdots,a_n \in \mathcal{A}} \sup_{x \in \mathcal{Z}_{\mathcal{A}}} e^{S_n\varphi([a_1,\cdots,a_n+x])}, \tag{2.8}$$

where $S_n\varphi(x)$ denotes the ergodic sum $\varphi(x) + \cdots + \varphi(T^{n-1}(x))$. For $\mathcal{A} = \mathbb{N}$ we denote $\mathsf{P}_{\mathbb{N}}(T, \varphi)$ by $\mathsf{P}(T, \varphi)$.

The nth variation of a function φ, denoted by $\mathrm{Var}_n(\varphi)$, is defined as

$$\mathrm{Var}_n(\varphi) := \sup_{x,z\,:\,I_n(x)=I_n(z)} |\varphi(x) - \varphi(z)|,$$

where $I_n(x)$ represents the basic cylinder of order n.

A function φ is said to satisfies the *tempered distortion property* if

$$\mathrm{Var}_1(\varphi) < \infty \text{ and } \lim_{n\to\infty} \mathrm{Var}_n(\varphi) = 0. \tag{2.9}$$

Throughout the thesis, we consider potential $\varphi : [0,1) \to \mathbb{R}$ to be a function satisfying the tempered distortion property. Since φ satisfy the tempered distortion property (2.9) it follows that removing the supremum from (2.8) will not effect the value of $\mathsf{P}_{\mathcal{A}}(T, \varphi)$.

The existence of the limit in (2.8) is due to the following result.

Proposition 2.2.1 (Walters, [44]) *The limit defining $\mathsf{P}_{\mathcal{A}}(T, \varphi)$ exists. Moreover if $\varphi : [0,1) \to \mathbb{R}$ is a function satisfying (2.9), then the value of $\mathsf{P}_{\mathcal{A}}(T, \varphi)$ remains unchanged even without taking the supremum over $x \in \mathcal{Z}_{\mathcal{A}}$ in (2.8).* ■

To avoid confusion, from now on, if we want to take any point z from the basic cylinder $I_n(a_1, \ldots, a_n)$, we can always take it as $z = \frac{p_n}{q_n} = [a_1, \ldots, a_n]$.

As all the potentials in this thesis satisfies (2.9), by Proposition 2.2.1, we can present (2.8) as

$$\mathsf{P}_{\mathcal{A}}(T,\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\cdots,a_n\in\mathcal{A}} e^{S_n\varphi([a_1,\cdots,a_n])}. \tag{2.10}$$

The next result which is by Hanus–Mauldin–Urbański [23] shows that when the Gauss system $([0,1),T)$ is approximated by its subsystems $(\mathcal{Z}_{\mathcal{A}},T)$ then in the system of continued fractions the pressure function has a continuity property (for an elementary proof see [38] or [23, Proposition 2]).

Proposition 2.2.2 (Hanus–Mauldin–Urbański, [23]) *Let $\varphi : [0,1) \to \mathbb{R}$ be a real function satisfying* (2.9)*. Then*

$$\mathsf{P}(T,\varphi) = \sup\{\mathsf{P}_{\mathcal{A}}(T,\varphi) : \mathcal{A} \text{ is a finite subset of } \mathbb{N}\}. \qquad \blacksquare$$

Notation: To simplify the presentation, we start by fixing some notation. We use $a \gg b$ to indicate that $|a/b|$ is sufficiently large, and $a \asymp b$ to indicate that $|a/b|$ is bounded between unspecified positive constants. We can also use *i.m.* for *infinitely many.*

Chapter 3

Metrical theory associated with the set $\mathcal{D}(\psi)^c$

As discussed in Chapter 1, most metric theories are intended to strengthen the asymptotic Diophantine approximation result (Corollary 1.1.3) instead of uniform Diophantine approximation result (Theorem 1.1.2). In this chapter, the Lebesgue measure criterion for the set $\mathcal{D}(\psi)^c$ which is attributed to Kleinbock–Wadleigh [34] and the Hausdorff measure of the set $\mathcal{D}(\psi)^c$ which is due to Hussain–Kleinbock–Wadleigh–Wang [27] are discussed in detail.

Throughout the thesis, for the decreasing function $\psi : [t_0, \infty) \to \mathbb{R}^+$ with $t_0 \geq 1$ fixed, the functions ψ and Ψ will always be related by the following auxiliary function

$$\Psi(t) := \frac{1}{1 - t\psi(t)} - 1. \tag{3.1}$$

3.1 Lebesgue measure of the set $\mathcal{D}(\psi)^c$

The contents in this section are mostly taken from [34].

From Theorem 1.1.2 it is obvious that $\mathcal{D}(\frac{1}{t}) = \mathbb{R}$. Also since $|q| < t$ in (1.18), therefore $\mathcal{D}(\psi) \subset \mathcal{W}(\psi)$ whenever ψ is decreasing. However $\mathcal{D}(\psi)$ is considerably different than $\mathcal{W}(\psi)$ for functions ψ which are decaying faster than $\frac{1}{t}$.

As noticed by Davenport–Schmidt [13], for any $\delta > 0$, the set $\mathcal{D}\left(\frac{1-\delta}{t}\right)$ is a subset of the union of $\mathbb{Q}$ and the set of badly approximable numbers. Thus

$$\lambda\left(\mathcal{D}\left(\frac{1-\delta}{t}\right)\right) = 0$$

and consequently

$$\lambda\left(\mathcal{D}\left(\frac{1-\delta}{t}\right)^c\right) = 1.$$

It is worth mentioning that even prior to Davenport–Schmidt's work, in regards to improving Dirichlet's Theorem, there were some contributions made by Diviš in the papers [14, 15] and some are made recently by Haas [22]. But beyond some particular choices of ψ, nothing was known until when Kleinbock–Wadleigh [34] proved a dichotomy law on the Lebesgue measure of the set $\mathcal{D}(\psi)^c$. Since then this set has gained much attention from many researchers. As observed by Kleinbock–Wadleigh [34], the set of ψ-Dirichlet non-improvable numbers is non-empty whenever ψ is decreasing. In fact, the starting point is the observation that Dirichlet's Theorem is sharp in the sense that if $\psi(t) < 1/t$ for all sufficiently large t, then there exists $x \in \mathbb{R}$ which is not ψ-Dirichlet improvable.

Kleinbock–Wadleigh [34, Lemma 2.1] noted by a straightforward proof that a real number x is ψ-Dirichlet improvable if and only if

$$|q_{n-1}x - p_{n-1}| < \psi(q_n) \tag{3.2}$$

for sufficiently large n. On the other hand, Cassels [11, §II.2] considered complete quotients of the form $\theta_{n+1} = [a_{n+1}(x), a_{n+2}(x), \dots]$ and $\phi_n = [a_n(x), a_{n-1}(x), \dots, a_1(x)]$ and derived the following beautiful relation

$$(1 + \theta_{n+1}\phi_n)^{-1} = q_n|q_{n-1}x - p_{n-1}|. \tag{3.3}$$

By combining the relation (3.3) with the ψ-Dirichlet property (3.2) of x, Kleinbock–Wadleigh proved the following important ψ-Dirichlet improvability criterion which, in other words, rephrases the ψ-Dirichlet improvability of x in terms of the growth of product of consecutive partial quotients.

Lemma 3.1.1 (Kleinbock–Wadleigh, [34]) Let $x \in [0, 1) \setminus \mathbb{Q}$ and $\psi : [t_0, \infty) \to \mathbb{R}^+$ with $t\psi(t) < 1$ for all $t \geq t_0$. Then

(i) $x \in \mathcal{D}(\psi)$ if $a_{n+1}(x)a_n(x) \leq \frac{\Psi(q_n)}{4}$ for all sufficiently large n,

(ii) $x \in \mathcal{D}(\psi)^c$ if $a_{n+1}(x)a_n(x) > \Psi(q_n)$ for infinitely many n. ■

From the above lemma it is obvious that an irrational number x satisfies the condition of ψ-Dirichlet improvability if and only if the product of two consecutive partial quotients of x do not grow rapidly.

Consequently, Kleinbock–Wadleigh [34] proved the following zero-one law for the Lebesgue measure of the set $\mathcal{D}(\psi)^c$.

Theorem 3.1.2 (Kleinbock–Wadleigh, [34]) *Let $\psi : [t_0, \infty) \to \mathbb{R}^+$ be a decreasing function and Ψ as defined in (3.3) (i.e., the function $t \mapsto t\psi(t)$ is increasing) with $t\psi(t) < 1$ for all $t \geq t_0$. If*

$$\sum_{t=t_0}^{\infty} \frac{\log \Psi(t)}{t\Psi(t)} < \infty \quad (resp. = \infty)$$

then

$$\lambda(\mathcal{D}(\psi)^c) = 0 \quad (\textit{resp. } \lambda(\mathcal{D}(\psi)) = 0).$$

Example 3.1.3

$$\lambda(\mathcal{D}(\psi)^c) = \begin{cases} 0, & \text{if} \quad \psi(t) = \frac{1}{t}\Big(1 - \frac{1}{\log t(\log\log t)^{2+\delta}}\Big) \text{ for any } \delta > 0; \\ \text{full}, & \text{if} \quad \psi(t) = \frac{1}{t}\Big(1 - \frac{1}{\log t(\log\log t)^{2}}\Big). \end{cases}$$

3.2 Hausdorff measure of the set $\mathcal{D}(\psi)^c$

Note that for ψ decreasing sufficiently slow, Theorem 3.1.2 tells that $\lambda(\mathcal{D}(\psi)^c) = 0$ but gives no further information about the size of these null sets. Keeping in view the importance of Hausdorff measure and dimension as discussed in Chapter 1, Hussain–Kleinbock–Wadleigh–Wang [27] have established the Hausdorff measure theoretic results for Dirichlet's improvability and they showed that the Hausdorff measure of $\mathcal{D}(\psi)^c$ satisfy a zero-infinity law for a wide range of dimension functions.

Throughout this section the results and material have been taken from [27]. For completeness we will include the proofs discussed in [27].

Theorem 3.2.1 (Hussain–Kleinbock–Wadleigh–Wang, [27]) *Let* $\psi : [t_0, \infty) \to \mathbb{R}^+$ *be a decreasing function with* $t\psi(t) < 1$ *for all* $t \geq t_0$*. Then for any* $s \in [0, 1)$

$$\mathcal{H}^s(\mathcal{D}(\psi)^c) = \begin{cases} 0 & \text{if} \quad \sum\limits_{t=t_0}^{\infty} t\left(\frac{1}{t^2\Psi(t)}\right)^s < \infty; \\ \infty & \text{if} \quad \sum\limits_{t=t_0}^{\infty} t\left(\frac{1}{t^2\Psi(t)}\right)^s = \infty. \end{cases}$$

Note that if we take $s = 1$ than this is the scope of Theorem 3.1.2 since $\mathcal{H}^1$ is the Lebesgue measure, therefore Hussain–Kleinbock–Wadleigh–Wang [27] pointed that the condition $s < 1$ is necessary.

Recall from Chapter 1, the f-dimensional Hausdorff measure $\mathcal{H}^f$ is a generalisation of the s-dimensional Hausdorff measure $\mathcal{H}^s$. A dimension function f is said to be *essentially sub-linear* if there exists $C > 1$ such that

$$\limsup_{x\to 0} \frac{f(Cx)}{f(x)} < C. \tag{3.4}$$

It was pointed out in [27] that condition (3.4) fails for $f(x) = x$ or $f(x) = x\log(1/x)$. Whereas it holds for $f(x) = x^s$ when $0 \leq s < 1$.

The following theorem implies Theorem 3.2.1.

Theorem 3.2.2 (Hussain–Kleinbock–Wadleigh–Wang, [27]) *Let ψ be as defined in Theorem 3.2.1 and let f be an essentially sub-linear dimension function. Then*

$$\mathcal{H}^f\left(\mathcal{D}(\psi)^c\right) = \begin{cases} 0 & \text{if} \quad \sum\limits_{t=t_0}^{\infty} tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty & \text{if} \quad \sum\limits_{t=t_0}^{\infty} tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

It is noteworthy that in [27], as a consequence of Lemma 3.1.1, the following significant inclusion was observed

$$G(\Psi) \subset \mathcal{D}(\psi)^c \subset G\left(\frac{\Psi}{4}\right), \tag{3.5}$$

where

$$G(\Psi) \quad = \quad \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\Big\},$$

which served as the basis for the proof of Theorem 3.2.2. Thus as a result of inclusion (3.5) the proof of the following theorem, implies the proof of Theorem 3.2.2.

Theorem 3.2.3 (Hussain–Kleinbock–Wadleigh–Wang, [27]) *Let $\psi : [t_0, \infty) \to \mathbb{R}^+$ be a decreasing function with $t\psi(t) < 1$ for all $t \geq t_0$ and Ψ satisfying relation (3.1). Also let f be an essentially sub-linear dimension function. Then*

$$\mathcal{H}^f\left(G(\Psi)\right) = \begin{cases} 0 & \text{if} \quad \sum\limits_{t} tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty & \text{if} \quad \sum\limits_{t} tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

Consider a set

$$\mathcal{K}(\Psi) := \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^2\Psi(q)} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

Just to avoid confusion, note that the set $\mathcal{K}(\Psi)$ is just the set of ψ-well approximable numbers (defined in Chapter 1) if we take $\psi(q) = \frac{1}{q\Psi(q)}$.

The following refined form of Jarník's Theorem was the key ingredient for proving Theorem 3.2.3 as discussed in [27].

Theorem 3.2.4 (Hussain–Kleinbock–Wadleigh–Wang, [27]) *Let Ψ be as defined above and let f be a dimension function satisfying the following conditions:*

$$\lim_{x\to 0} \frac{f(x)}{x} = \infty, \tag{3.6}$$

and

$$\exists\, C \geq 1 \text{ such that } \frac{f(x_2)}{x_2} \leq C\frac{f(x_1)}{x_1} \text{ whenever } x_1 < x_2 \ll 1. \tag{3.7}$$

Then

$$\mathcal{H}^f\left(\mathcal{K}(\Psi)\right) = \begin{cases} 0 & \text{if} \quad \sum\limits_{t=t_0}^{\infty} tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty & \text{if} \quad \sum\limits_{t=t_0}^{\infty} tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

It is worth mentioning here that for proving the divergence case of Theorem 3.2.3, Hussain–Kleinbock–Wadleigh–Wang [27] observed an obvious inclusion

$$G(\Psi) \supset \{x \in [0,1) : a_{n+1}(x) > \Psi(q_n) \text{ for i.m. } n \in \mathbb{N}\} =: G_1(\Psi).$$

Further they assumed that $\Psi(t) \geq 1$ for all $t \gg 1$. Otherwise, $\Psi(t) < 1$ for all t large enough since it is assumed that Ψ is non-decreasing. Then clearly $G_1(\Psi)$ and consequently $G(\Psi)$ consists of all irrationals in $[0,1]$, and that the sum in Theorem 3.2.3 diverges.

Additionally, they noticed that $\mathcal{K}(3\Psi)$ is properly contained in $G_1(\Psi)$. For the requirement of this thesis we will rewrite the proof from [27] for completeness.

In fact, if there are infinitely many pairs (p,q) with

$$\left|x - \frac{p}{q}\right| < \frac{1}{3\Psi(q)q^2} < \frac{1}{2q^2},$$

then by Legendre's Theorem (1.4) and by the monotonicity of Ψ,

$$\left|x - \frac{p_n}{q_n}\right| = \left|x - \frac{p}{q}\right| < \frac{1}{3\Psi(q)q^2} \leq \frac{1}{3\Psi(q_n)q_n^2}.$$

On the other hand by (2.6),

$$\left|x - \frac{p_n}{q_n}\right| \geq \frac{1}{3a_{n+1}q_n^2}.$$

Thus, for infinitely many n

$$a_{n+1} > \Psi(q_n),$$

verifying the claim.

Thus by Theorem 3.2.4 one will have

$$\mathcal{H}^f\left(G(\Psi)\right) \geq \mathcal{H}^f\left(G_1(\Psi)\right) \geq \mathcal{H}^f\left(\mathcal{K}(3\Psi)\right) = \infty,$$

whenever one can show that the dimension function f satisfies conditions (3.6) and (3.7), and that

$$\sum_{t=t_0}^{\infty} tf\left(\frac{1}{3t^2\Psi(t)}\right) = \infty.$$

This is done via the following lemma.

Lemma 3.2.5 (Hussain–Kleinbock–Wadleigh–Wang, [27]) Let f be an essentially sub-linear dimension function. Then both (3.6) and (3.7) hold. ■

3.3 Hausdorff dimension of the set $\mathcal{D}(\psi)^c$

In the previous sections we have mentioned measure theoretic results related with the set $\mathcal{D}(\psi)^c$ which will be helpful for proving the main results of this thesis. The Hausdorff dimension of the set $\mathcal{D}(\psi)^c$ follows directly from Theorem 3.2.1 as shown by Hussain–Kleinbock–Wadleigh–Wang [27] as follows.

$$\dim_{\mathrm{H}} \mathcal{D}(\psi)^c = \frac{2}{2+\tau}, \text{ where } \tau = \liminf_{t\to\infty} \frac{\log \Psi(t)}{\log t}.$$

As an example,

$$\dim_{\mathrm{H}} \mathcal{D}(\psi)^c = \frac{2}{2+\tau}, \text{ for } \psi(t) = \frac{1-at^{-\tau}}{t} \ (a>0, \tau>0).$$

3.4 Natural Question

From the discussion in Section 3.2, we know that

$$\mathcal{K}(3\Psi) \subset G(\Psi). \tag{3.8}$$

If we consider the the s-dimensional Hausdorff measure statement for the set $\mathcal{K}(\Psi)$ where $s \in (0,1)$. Then

$$\mathcal{H}^s(\mathcal{K}(\Psi)) = \begin{cases} 0 & \text{if } \sum\limits_{t=t_0}^{\infty} t\left(\frac{1}{t^2\Psi(t)}\right)^s < \infty; \\ \infty & \text{if } \sum\limits_{t=t_0}^{\infty} t\left(\frac{1}{t^2\Psi(t)}\right)^s = \infty, \end{cases}$$

readily gives the divergence statement for $\mathcal{K}(3\Psi)$. To be precise,

$$\mathcal{H}^s(\mathcal{K}(3\Psi)) = \infty \Longrightarrow \mathcal{H}^s(G(\Psi)) = \infty.$$

It is thus clear that when the sum $\sum\limits_{t=t_0}^{\infty} t\left(\frac{1}{t^2\Psi(t)}\right)^s$ diverges, both the sets $G(\Psi)$ and $\mathcal{K}(3\Psi)$ have full measure. However, since the inclusion (3.8) is proper, it is natural to expect that the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$ is non-trivial. From a measure theoretic point of view there is no new information, however, from a dimension point of view there is more to ask.
This raises the natural question.

Question 3.4.1 How big is the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$? ■

In the next chapter we answer the above question. To be more specific, using a Cantor-type construction and the mass distribution principle we will determine the Hausdorff dimension of the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$ for any $C > 0$. We will that this set is uncountable.

Chapter 4

Well approximable versus Dirichlet improvable numbers

In this chapter we aim to answer Question 3.4.1.

4.1 Statement of the main result

Recall that

$$G(\Psi) = \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N}\Big\}$$

and

$$\mathcal{K}(\Psi) := \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^2\Psi(q)} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

We will completely determine the Hausdorff dimension for the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$ for any $C > 0$.

Theorem 4.1.1 *Let $\Psi : [1, \infty) \to \mathbb{R}^+$ be a non-decreasing function and $C > 0$. Then*

$$\dim_{\mathrm{H}} \Big(G(\Psi) \setminus \mathcal{K}(C\Psi)\Big) = \frac{2}{\tau + 2} = \dim_{\mathrm{H}} G(\Psi), \ \ \textit{where } \tau = \liminf_{q\to\infty} \frac{\log \Psi(q)}{\log q}.$$

∎

The term τ gives information regarding how a function Ψ grows near infinity and is known as the *lower order at infinity*. It appears naturally in determining the Hausdorff dimension of exceptional sets, when general distance functions are involved, see [16, 17].

Recall from Chapter 2, the process to obtain the Hausdorff dimension of a set normally consists of finding the upper and the lower bounds separately. Therefore we will divide the proof of Theorem 4.1.1 into two parts: **the upper bound** and **the lower bound**.

4.2 Proof of Theorem 4.1.1: the upper bound

For ease of calculations, we choose $C = 1$ throughout the remainder of this chapter.

As

$$(G(\Psi) \setminus \mathcal{K}(\Psi)) \subseteq G(\Psi) \subset \mathcal{D}(\psi)^c.$$

Therefore the upper bound for the Hausdorff dimension of the set $G(\Psi) \setminus \mathcal{K}(\Psi)$ follows directly from the $\dim_{\mathrm{H}} G(\Psi)$, thus from Section 3.3, we have

$$\dim_{\mathrm{H}} \Big(G(\Psi) \setminus \mathcal{K}(\Psi) \Big) \leq \dim_{\mathrm{H}} G(\Psi) \ \leq \frac{2}{\tau + 2}.$$

Thus the proof of Theorem 4.1.1 follows from establishing the complementary lower bound.

4.3 Proof of Theorem 4.1.1: the lower bound

Notice that the set $E := G(\Psi) \setminus \mathcal{K}(\Psi)$ can be written as

$$E = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Psi(q_n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

To illustrate the main ideas, we first prove the result for a specific choice of the approximating function $\Psi(q_n) := q_n^\tau$ for any $\tau > 0$. Proving the result for the general approximating function $\Psi(q_n)$ instead of q_n^τ will require a slight modification to the arguments presented below but essentially the process is the same. We will briefly sketch this process in the last section.

The set E can now be written as

$$E = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq q_n^\tau \ \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < q_n^\tau \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

We aim to show that

$$\dim_{\mathrm{H}} E \geq \frac{2}{\tau + 2}.$$

Fix a large integer L, and define $S = S(L, M)$ to be the solution to the equation

$$\sum_{\substack{1 \leq a_i \leq M \\ 1 \leq i \leq L}} \left(\frac{1}{q_L^{2+\tau}(a_1, \cdots, a_L)} \right)^S = 1. \tag{4.1}$$

It follows from the definition of the pressure function, as $L, M \to \infty$, that $S \to \frac{2}{2+\tau}$. The process of proving this follows as in [45, Lemma 2.6], therefore we skip it. So, it remains to show that

$$\dim_{\mathrm{H}} E \geq S.$$

As discussed in Section 2.1 the main strategy in obtaining the lower bound is to use the mass distribution principle i.e., Proposition 2.1.4. To employ it, we systematically divide the process into the following subsections.

Cantor subset construction

Choose a rapidly increasing sequence of integers $\{n_k\}_{k\geq 1}$ such that $n_k \gg n_{k-1}$, for all k. For convenience define $n_0 = 0$.

Define the subset $\mathcal{E}_M$ of E as follows

$$\mathcal{E}_M = \left\{ x \in [0,1) : \begin{array}{c} \frac{1}{4}q^{\tau}_{n_k-1} \leq a_{n_k}(x) \leq \frac{1}{2}q^{\tau}_{n_k-1} \text{ and } a_{n_k-1}(x) = 4 \\ \text{and } 1 \leq a_j(x) \leq M, \text{ for all } j \neq n_k - 1, n_k \end{array} \right\}.$$

For any $n \geq 1$, define strings $(a_1, \ldots, a_n)$ by

$$D_n = \left\{ (a_1, \ldots, a_n) \in \mathbb{N}^n : \begin{array}{c} \frac{1}{4}q^{\tau}_{n_k-1} \leq a_{n_k}(x) \leq \frac{1}{2}q^{\tau}_{n_k-1} \text{ and } a_{n_k-1}(x) = 4 \\ \text{and } 1 \leq a_j(x) \leq M, \text{ for all } 1 \leq j \neq n_k - 1, n_k \leq n \end{array} \right\}.$$

For any $n \geq 1$ and $(a_1, \ldots, a_n) \in D_n$, we call $I_n(a_1, \ldots, a_n)$ a *basic cylinder of order* n and

$$J_n := J_n(a_1, \ldots, a_n) := \bigcup_{a_{n+1}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}) \tag{4.2}$$

a *fundamental cylinder of order* n, where the union in (4.2) is taken over all a_{n+1} such that $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}$.

Summary: We will consider three distinct cases for J_n according to the limitations on the partial quotients. The following table (commencing from $k = 1$), summarises our Cantor set construction such that for $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}$:

$$\begin{array}{ll} n_k \leq n \leq n_{k+1} - 3, & J_n = \bigcup_{1 \leq a_{n+1}(x) \leq M} I_{n+1}(a_1, \ldots, a_n, a_{n+1}), \\ n = n_{k+1} - 2, & J_n = I_{n+1}(a_1, \ldots, a_n, 4), \\ n = n_{k+1} - 1, & J_n = \bigcup_{\frac{1}{4}q_n^{\tau} \leq a_{n+1}(x) \leq \frac{1}{2}q_n^{\tau}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}). \end{array}$$

It is now clear that

$$\mathcal{E}_M = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \ldots, a_n) \in D_n} J_n(a_1, \ldots, a_n).$$

Lengths of fundamental intervals

We now calculate lengths of fundamental cylinders split into three distinct cases, following from the construction of $\mathcal{E}_M$ and the definition of fundamental cylinders.

Case I. When $n_k \leq n \leq n_{k+1} - 3$ for any $k \geq 1$, we have

$$J_n(a_1, \ldots, a_n) = \bigcup_{1 \leq a_{n+1}(x) \leq M} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$$

Therefore,

$$|J_n(a_1,\ldots,a_n)| = \frac{M}{(q_n+q_{n-1})((M+1)q_n+q_{n-1})} \tag{4.3}$$

and

$$\frac{1}{6q_n^2} \leq |J_n(a_1,\ldots,a_n)| \leq \frac{1}{q_n^2}.$$

In particular for $n=n_k$, we have $\frac{1}{4}q_{n-1}^{\tau} \leq a_n(x) \leq \frac{1}{2}q_{n-1}^{\tau}$. Therefore,

$$|J_n(a_1,\ldots,a_n)| \leq \frac{1}{q_n^2} = \frac{1}{(a_nq_{n-1}+q_{n-2})^2} \leq \frac{1}{(a_nq_{n-1})^2} = \frac{1}{\frac{1}{16}q_{n-1}^{2+2\tau}},$$

and

$$|J_n(a_1,\ldots,a_n)| \geq \frac{1}{6q_n^2} = \frac{1}{6(a_nq_{n-1}+q_{n-2})^2} \geq \frac{1}{\frac{3}{2}q_{n-1}^{2+2\tau}}.$$

Therefore for $n=n_k$, we have

$$\frac{1}{\frac{3}{2}q_{n-1}^{2+2\tau}} \leq |J_n(a_1,\ldots,a_n)| \leq \frac{1}{\frac{1}{16}q_{n-1}^{2+2\tau}}.$$

Case II. When $n=n_{k+1}-2$, we have

$$J_n = I_n(a_1,\ldots,a_n,4).$$

Therefore,

$$|J_n(a_1,\ldots,a_n)| = \frac{1}{(4q_n+q_{n-1})(5q_n+q_{n-1})} \tag{4.4}$$

and

$$\frac{1}{60q_n^2} \leq |J_n(a_1,\ldots,a_n)| \leq \frac{1}{16q_n^2}.$$

Case III. When $n=n_{k+1}-1$, we have

$$J_n = \bigcup_{\frac{1}{4}q_n^{\tau} \leq a_{n+1}(x) \leq \frac{1}{2}q_n^{\tau}} I_{n+1}\left(a_1,\ldots,a_n,a_{n+1}\right).$$

Therefore,

$$|J_n(a_1,\ldots,a_n)| = \frac{\frac{1}{4}q_n^{\tau}+1}{(\frac{1}{4}q_n^{\tau+1}+q_{n-1})(\frac{1}{2}q_n^{\tau+1}+q_n+q_{n-1})} \tag{4.5}$$

and

$$\frac{1}{\frac{3}{2}q_n^{2+\tau}} \leq |J_n(a_1,\ldots,a_n)| \leq \frac{1}{\frac{1}{4}q_n^{2+\tau}}.$$

Gap estimation

In this section we estimate the gap between $J_n(a_1, \dots, a_n)$ and its adjoint fundamental cylinder of the same order n. These gaps are helpful for estimating the measure on general balls.

Let $J_{n-1}(a_1, \dots, a_{n-1})$ be the mother fundamental cylinder of $J_n(a_1, \dots, a_n)$. Without loss of generality, assume that n is even, since if n is odd we can carry out the estimation in almost the same way. Let the left and the right gap between $J_n(a_1, \dots, a_n)$ and its adjoint fundamental cylinder at each side be represented by $g_n^{\ell}(a_1, \dots, a_n)$ and $g_n^{r}(a_1, \dots, a_n)$ respectively.

Denote by $g_n(a_1, \dots, a_n)$ the minimum distance between $J_n(a_1, \dots, a_n)$ and its adjacent cylinder of the same order n, that is,

$$g_n(a_1, \dots, a_n) = \min\{g_n^{\ell}(a_1, \dots, a_n), g_n^{r}(a_1, \dots, a_n)\}.$$

Since n is even, the right adjoint fundamental cylinder to J_n, which is contained in J_{n-1}, is

$$J_n' = J_n(a_1, \dots, a_{n-1}, a_n + 1) \text{ (if it exists)}$$

and the left adjoint fundamental cylinder to J_n, which is contained in J_{n-1}, is

$$J_n'' = J_n(a_1, \dots, a_{n-1}, a_n - 1) \text{ (if it exists).}$$

We distinguish three cases according to the range of n defined for $\mathcal{E}_M$. The estimation is based on the distribution of cylinders, as described in the summary in Section 4.3.

Gap I. For the case $n_k \le n \le n_{k+1} - 3$, we have

$$\begin{aligned}
J_n &= \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}\left(a_1, \dots, a_n, a_{n+1}\right), \\
J_n' &= \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}\left(a_1, \dots, a_n + 1, a_{n+1}\right), \\
J_n'' &= \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}\left(a_1, \dots, a_n - 1, a_{n+1}\right).
\end{aligned}$$

Then by Proposition 2.1.2, for the right gap

$$g_n^{r}(a_1, \dots, a_n) \ge \frac{1}{(q_n + q_{n-1})((M+1)(q_n + q_{n-1}) + q_{n-1})}$$

and for the left gap

$$g_n^{l}(a_1, \dots, a_n) \ge \frac{1}{q_n((M+1)q_n + q_{n-1})}.$$

So

$$g_n(a_1, \dots, a_n) = \frac{1}{(q_n + q_{n-1})((M+1)(q_n + q_{n-1}) + q_{n-1})}. \tag{4.6}$$

Also from (4.3) and (4.6) we have

$$g_n(a_1,\ldots,a_n) \geq \frac{1}{2M}|J_n(a_1,\ldots,a_n)|.$$

Gap II. For the case $n = n_{k+1}-2$, we have

$$\begin{aligned}
J_n &= I_{n+1}(a_1,\ldots,a_n,4) \subset I_n(a_1,\ldots,a_n),\\
J'_n &= I_{n+1}(a_1,\ldots,a_n+1,4) \subset I_n(a_1,\ldots,a_n+1),\\
J''_n &= I_{n+1}(a_1,\ldots,a_n-1,4) \subset I_n(a_1,\ldots,a_n-1).
\end{aligned}$$

Since J_n lies in the middle of $I_n(a_1,\ldots,a_n)$ and J'_n lies on the right to $I_n(a_1,\ldots,a_n)$ therefore the right gap is larger than the distance between the right endpoint of J_n and that of I_n. Also, as J''_n lies on the left to $I_n(a_1,\ldots,a_n)$ therefore the left gap is larger than the distance between the left endpoint of J_n and that of I_n.

Hence, for the right gap

$$g_n^r(a_1,\ldots,a_n) \geq \frac{p_n+p_{n-1}}{q_n+q_{n-1}} - \frac{4p_n+p_{n-1}}{4q_n+q_{n-1}} = \frac{3}{(q_n+q_{n-1})(4q_n+q_{n-1})}.$$

and for the left gap

$$g_n^l(a_1,\ldots,a_n) \geq \frac{5p_n+p_{n-1}}{5q_n+q_{n-1}} - \frac{p_n}{q_n} = \frac{1}{(5q_n+q_{n-1})q_n}.$$

Therefore,

$$g_n(a_1,\ldots,a_n) \geq \frac{1}{(5q_n+q_{n-1})(q_n+q_{n-1})}. \tag{4.7}$$

Also, from (4.4) and (4.7) we, we notice that

$$g_n(a_1,\ldots,a_n) \geq \frac{4}{3}|J_n(a_1,\ldots,a_n)|.$$

Gap III. For the case $n = n_{k+1}-1$, we have

$$\begin{aligned}
J_n &= \bigcup_{\frac{1}{4}q_n^\tau \leq a_{n+1}(x) \leq \frac{1}{2}q_n^\tau} I_{n+1}(a_1,\ldots,a_n,a_{n+1}),\\
J'_n &= \bigcup_{\frac{1}{4}q_n^\tau \leq a_{n+1}(x) \leq \frac{1}{2}q_n^\tau} I_{n+1}(a_1,\ldots,a_n+1,a_{n+1}),\\
J''_n &= \bigcup_{\frac{1}{4}q_n^\tau \leq a_{n+1}(x) \leq \frac{1}{2}q_n^\tau} I_{n+1}(a_1,\ldots,a_n-1,a_{n+1}).
\end{aligned}$$

In this case also the gap position description is the same as the case when $n = n_{k+1}-2$.

Hence, for the right gap

$$g_n^r(a_1,\ldots,a_n) \geq \frac{(\frac{1}{4}q_n^\tau - 1)}{(\frac{1}{4}q_n^\tau q_n + q_{n-1})(q_n+q_{n-1})},$$

and for the left gap

$$g_n^l(a_1,\dots,a_n) \geq \frac{1}{((\frac{1}{2}q_n^\tau+1)q_n+q_{n-1})q_n}.$$

Therefore,

$$g_n(a_1,\dots,a_n) \geq \frac{1}{((\frac{1}{2}q_n^\tau+1)q_n+q_{n-1})(q_n+q_{n-1})}. \tag{4.8}$$

By comparing (4.5) with (4.8), we obtain

$$g_n(a_1,\dots,a_n) \geq \frac{1}{3}|J_n(a_1,\dots,a_n)|.$$

Mass Distribution on $\mathcal{E}_M$

We define a measure μ supported on $\mathcal{E}_M$. For this we start by defining the measure on the fundamental cylinders of order n_k-2, n_k-1 and n_k. The measure on other fundamental cylinders can be obtained by using the consistency of a measure. Because the sparse set $\{n_k\}_{k\geq 1}$ is of our choosing, we may let $m_{k+1}L = n_{k+1}-2-n_k$ for any $k\geq 0$. This simplifies calculations without loss of generality.

Note that the sum in (4.1) induces a measure μ on a basic cylinder of order L

$$\mu(I_L(a_1,\dots,a_L)) = \left(\frac{1}{q_L^{2+\tau}}\right)^S,$$

for each $1\leq a_1,\dots,a_L\leq M$.

Step I. Let $1\leq i\leq m_1$. We first define a positive measure for the fundamental cylinders $J_{iL}(a_1,\dots,a_{iL})$ i.e.,

$$\mu\left(J_{iL}(a_1,\dots,a_{iL})\right) = \prod_{t=0}^{i-1}\left(\frac{1}{q_L^{2+\tau}(a_{tL+1},\dots,a_{(t+1)L})}\right)^S$$

and then we distribute this measure uniformly over its next offspring.

Step II. For J_{n_1-1} and J_{n_1-2}, define a measure

$$\begin{aligned}\mu\left(J_{n_1-1}(a_1,\dots,a_{n_1-1})\right) &= \mu\left(J_{n_1-2}(a_1,\dots,a_{n_1-2})\right)\\ &= \prod_{t=0}^{m_1-1}\left(\frac{1}{q_L^{2+\tau}(a_{tL+1},\dots,a_{(t+1)L})}\right)^S.\end{aligned}$$

Step III. For J_{n_1}, define a measure

$$\mu\left(J_{n_1}(a_1,\dots,a_{n_1})\right) = \frac{1}{\frac{1}{4}q_{n_1-1}^\tau}\mu\left(J_{n_1-1}(a_1,\dots,a_{n_1-1}\right).$$

In other words, the measure of J_{n_1-1} is uniformly distributed on its next offspring J_{n_1}.

Measure of other levels. The measure of fundamental cylinders for other levels can be defined inductively.

To define the measure on general fundamental cylinders $J_{n_{k+1}-2}$ and $J_{n_{k+1}-1}$, we assume that $\mu\left(J_{n_k}\right)$ has been defined. Then define

$$\begin{aligned}\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right) &= \mu\left(J_{n_{k+1}-2}(a_1,\ldots,a_{n_{k+1}-2})\right)\\ &= \mu\left(J_{n_k}(a_1,\ldots,a_{n_k})\right)\cdot\prod_{t=0}^{m_{k+1}-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_k+tL+1},\ldots,a_{n_k+(t+1)L})}\right)^S.\end{aligned}$$

Next, we equally distribute the measure of the fundamental cylinder $J_{n_{k+1}-1}$ among its next offspring which is a fundamental cylinder of order n_{k+1} i.e.,

$$\mu\left(J_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})\right) = \frac{1}{\frac{1}{4}q^{\tau}_{n_{k+1}-1}}\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right).$$

The measure of other fundamental cylinders of level less than $n_{k+1}-2$, is given using the consistency of the measure. Therefore, for $n=n_k+iL$ where $1\le i\le m_{k+1}$, we define

$$\begin{aligned}\mu\left(J_{n_k+iL}(a_1,\ldots,a_{n_k+iL})\right) = \mu\left(J_{n_k}(a_1,\ldots,a_{n_k})\right)&\\ \cdot\prod_{t=0}^{i-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_k+tL+1},\ldots,a_{n_k+(t+1)L})}\right)^S&.\end{aligned}$$

The Hölder exponent of the measure μ

For the lower bound, in order to apply the mass distribution principle to the Cantor subset $\mathcal{E}_M$, we need the measure of a general ball. So far we have only calculated $\mu\left(J_n(a_1,\ldots,a_n)\right)$. We show that there is a Hölder condition between $\mu\left(J_n(a_1,\ldots,a_n)\right)$ and $|J_n(a_1,\ldots,a_n)|$ and another Hölder condition between the ball $\mu(B(x,r))$ and radius r. The derived inequalities continue the program of establishing our lower bound.

Estimation of $\mu\left(J_n(a_1,\ldots,a_n)\right)$

First, we estimate the Holder exponent of $\mu\left(J_n(a_1,\ldots,a_n)\right)$ in relation to $|J_n(a_1,\ldots,a_n)|$. In simple words we want to compare the measure of fundamental cylinders with their lengths.

Step I. When $n=iL$, for some $1\le i<m_1$,

$$\begin{aligned}\mu\left(J_{iL}\left(a_1,\ldots,a_{iL}\right)\right) &= \prod_{t=0}^{i-1}\left(\frac{1}{q_L^{2+\tau}(a_{tL+1},\ldots,a_{(t+1)L})}\right)^S\\ &\overset{(2.5)}{\le} 2^{(2+\tau)(i-1)}\left(\frac{1}{q_{iL}^{2+\tau}(a_1,\ldots,a_{iL})}\right)^S \end{aligned}\tag{4.9}$$

$$\overset{(2.4)}{\leq} \left(\frac{1}{q_{iL}^{2+\tau}(a_1, \ldots, a_{iL})}\right)^{S-2/L}$$
$$\ll |J_{iL}(a_1, \ldots, a_{iL})|^{S-2/L}.$$

Step II(a). When $n = m_1 L = n_1 - 2$,

$$\begin{aligned}
\mu\left(J_{n_1-2}(a_1, \ldots, a_{n_1-2})\right) &= \prod_{t=0}^{m_1-1} \left(\frac{1}{q_L^{2+\tau}(a_{tL+1}, \ldots, a_{(t+1)L})}\right)^S \\
&\overset{(4.9)}{\leq} 2^{(2+\tau)(m_1-1)} \left(\frac{1}{q_{m_1L}^{2+\tau}(a_1, \ldots, a_{m_1L})}\right)^S \\
&\leq 2^{(2+\tau)(m_1-1)} \left(\frac{1}{q_{n_1-2}^{2+\tau}(a_1, \ldots, a_{n_1-2})}\right)^S \\
&\leq \left(\frac{1}{q_{n_1-2}^{2+\tau}(a_1, \ldots, a_{n_1-2})}\right)^{S-\frac{2}{L}} \qquad (4.10) \\
&\ll |J_{n_1-2}(a_1, \ldots, a_{n_1-2})|^{S-2/L}.
\end{aligned}$$

Step II(b). When $n = n_1 - 1 = m_1 L + 1$,

$$\begin{aligned}
\mu\left(J_{n_1-1}(a_1, \ldots, a_{n_1-1})\right) &= \mu\left(J_{n_1-2}(a_1, \ldots, a_{n_1-2})\right) \\
&\overset{(4.10)}{\leq} \left(\frac{1}{q_{n_1-2}^{2+\tau}(a_1, \ldots, a_{n_1-2})}\right)^{S-\frac{2}{L}} \\
&\asymp \left(\frac{1}{q_{n_1-1}^{2+\tau}(a_1, \ldots, a_{n_1-1})}\right)^{S-\frac{2}{L}} \qquad (4.11) \\
&\leq c|J_{n_1-1}(a_1, \ldots, a_{n_1-1})|^{S-\frac{2}{L}},
\end{aligned}$$

where $c = \frac{3}{2}$ and inequality (4.11) is obtained from the relation

$$q_{n_{k+1}-1}(a_1, \ldots, a_{n_{k+1}-2}, 4) \asymp q_{n_{k+1}-2}(a_1, \ldots, a_{n_{k+1}-2})$$

defined for any k.

Step III. For $n = n_1$, using inequality (4.11), we have

$$\begin{aligned}
\mu\left(J_{n_1}(a_1, \ldots, a_{n_1})\right) &= \frac{1}{\frac{1}{4}q_{n_1-1}^{\tau}} \mu\left(J_{n_1-1}(a_1, \ldots, a_{n_1-1})\right) \\
&\leq \frac{1}{\frac{1}{4}q_{n_1-1}^{\tau}}\, c \left(\frac{1}{q_{n_1-1}^{2+\tau}(a_1, \ldots, a_{n_1-1})}\right)^{S-\frac{2}{L}} \\
&\leq \frac{1}{\frac{1}{4}}\, c \left(\frac{1}{q_{n_1-1}^{2+2\tau}(a_1, \ldots, a_{n_1-1})}\right)^{S-\frac{2}{L}} \\
&\ll |J_{n_1}(a_1, \ldots, a_{n_1})|^{S-\frac{2}{L}}.
\end{aligned}$$

Next we find Hölder exponent for the general fundamental cylinder $J_{n_{k+1}-1}$. The Hölder exponent for cylinders of other levels can be carried out in the same way.

Let $n = n_{n_{k+1}-1}$. Recall that

$$\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right) = \mu\left(J_{n_{k+1}-2}(a_1,\ldots,a_{n_{k+1}-2})\right)$$
$$= \left[\prod_{j=0}^{k-1}\left(\frac{1}{\frac{1}{4}q^{\tau}_{n_{j+1}-1}}\prod_{t=0}^{m_{j+1}-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_j+tL+1},\ldots,a_{n_j+(t+1)L})}\right)^S\right)\right]$$
$$\cdot\prod_{t=0}^{m_{k+1}-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_k+tL+1},\ldots,a_{n_k+(t+1)L})}\right)^S.$$

By arguments similar to **Step I** and **Step II**, we obtain

$$\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right) \le \prod_{j=0}^{k-1}\left(\frac{1}{\frac{1}{4}q^{\tau}_{n_{j+1}-1}}\left(\frac{1}{q^{2+\tau}_{m_{j+1}L}(a_{n_j+1},\ldots,a_{n_j+(m_{j+1})L})}\right)^{S-\frac{2}{L}}\right)$$
$$\cdot\left(\frac{1}{q^{2+\tau}_{m_{k+1}L}(a_{n_k+1},\ldots,a_{n_k+(m_{k+1})L})}\right)^{S-\frac{2}{L}}$$
$$\le 2^{2k}\cdot\left(\frac{1}{q^{2+\tau}_{n_{k+1}-2}}\right)^{S-\frac{6}{L}} \le \left(\frac{1}{q^{2+\tau}_{n_{k+1}-2}}\right)^{S-\frac{10}{L}}$$
$$\asymp \left(\frac{1}{q^{2+\tau}_{n_{k+1}-1}}\right)^{S-\frac{10}{L}}$$
$$\le c_3\left|J_{n_{k+1}-1}\right|^{S-\frac{10}{L}},$$

where $c_3 = \frac{3}{2}$. Here for the third inequality, we use

$$q^{2(2+\tau)}_{n_{k+1}-2} \ge q^2_{n_{k+1}-2} \ge 2^{n_{k+1}-3} \ge 2^{L(m_1+\ldots+m_{k+1})} \ge 2^{L(k+1)} \ge 2^{Lk} = 2^{2k\cdot\frac{L}{2}}.$$

Consequently,

$$\mu\left(J_{n_{k+1}}\left(a_1,\ldots,a_{n_{k+1}}\right)\right) = \frac{1}{\frac{1}{4}q^{\tau}_{n_{k+1}-1}}\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right)$$
$$\le \frac{1}{\frac{1}{4}}\left(\frac{1}{q^{2+2\tau}_{n_{k+1}-1}}\right)^{S-\frac{10}{L}}$$
$$\ll \left|J_{n_{k+1}}\left(a_1,\ldots,a_{n_{k+1}}\right)\right|^{S-\frac{10}{L}}.$$

In summary, we have shown that, for any $n \ge 1$ and $(a_1,\ldots,a_n)$,

$$\mu\left(J_n\left(a_1,\ldots,a_n\right)\right) \ll \left|J_n\left(a_1,\ldots,a_n\right)\right|^{S-\frac{10}{L}}.$$

Estimation of a general ball $\mu(B(x,r))$

Assume that $x \in \mathcal{E}_M$ and $B(x,r)$ is a ball centred at x with radius r small enough. For each $n \ge 1$, let $J_n = J_n(a_1,\ldots,a_n)$ contain x and

$$g_{n+1}(a_1,\ldots,a_{n+1}) \le r < g_n(a_1,\ldots,a_n).$$

Clearly, by the definition of g_n, we see that

$$B(x,r)\cap \mathcal{E}_M \subset J_n(a_1,\ldots,a_n).$$

Case I. $n=n_{k+1}-1$.

(i) $r\le |I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})|$. In this case the ball $B(x,r)$ can intersect at most four basic cylinders of order n_{k+1}, which are

$$\begin{aligned}&I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}-1), \quad I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}),\\ &I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}+1), \quad I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}+2).\end{aligned}$$

Thus we have

$$\begin{aligned}\mu(B(x,r)) &\le 4\mu(J_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}))\\ &\le 4c_0|J_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})|^{S-\frac{10}{L}}\\ &\le 8c_0 M g_{n_{k+1}}^{S-\frac{10}{L}}\\ &\le 8c_0 M r^{S-\frac{10}{L}}.\end{aligned}$$

(ii) $r>|I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})|$. In this case, since

$$|I_{n_k}(a_1,\ldots,a_{n_k})|=\frac{1}{q_{n_{k+1}}(q_{n_{k+1}}+q_{n_{k+1}-1})}\ge \frac{1}{2q_{n_{k+1}-1}^{2+2\tau}},$$

the number of fundamental cylinders of order n_{k+1} contained in $J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})$ that the ball $B(x,r)$ intersects is at most

$$4rq_{n_{k+1}-1}^{2+2\tau}+2\le 8rq_{n_{k+1}-1}^{2+2\tau}.$$

Thus we have

$$\begin{aligned}\mu(B(x,r)) &\le \min\left\{\mu(J_{n_{k+1}-1}), 8rq_{n_{k+1}-1}^{2\tau}q_{n_{k+1}-1}^{2}\mu(J_{n_{k+1}})\right\}\\ &\le \mu(J_{n_{k+1}-1})\min\left\{1, 8rq_{n_{k+1}-1}^{2\tau}q_{n_{k+1}-1}^{2}\frac{1}{q_{n_{k+1}-1}^{\tau}}\right\}\\ &\le c|J_{n_{k+1}-1}|^{S-\frac{10}{L}}\min\left\{1, 8rq_{n_{k+1}-1}^{2+\tau}\right\}\\ &\le c\left(\frac{1}{q_{n_{k+1}-1}^{2+\tau}}\right)^{S-\frac{10}{L}}\min\left\{1, 8rq_{n_{k+1}-1}^{2+\tau}\right\}\\ &\le c\left(\frac{1}{q_{n_{k+1}-1}^{2+\tau}}\right)^{S-\frac{10}{L}}(8rq_{n_{k+1}-1}^{2+\tau})^{S-\frac{10}{L}}\\ &\le Cr^{S-\frac{10}{L}}, \text{ where } C=c8^{S-\frac{10}{L}}.\end{aligned}$$

Here we use $\min\{a,b\}\le a^{1-s}b^s$ for any $a,b>0$ and $0\le s\le 1$.

Case II. $n = n_{k+1} - 2$.

For $r > |I_{n_{k+1}-1}(a_1, \ldots, a_{n_{k+1}-1})|$, since

$$|I_{n_{k+1}-1}(a_1, \ldots, a_{n_{k+1}-1})| \geq \frac{1}{128 q^2_{n_{k+1}-2}},$$

the number of fundamental cylinders of order $n_{k+1} - 1$ contained in $J_{n_{k+1}-2}(a_1, \ldots, a_{n_{k+1}-2})$ that the ball $B(x, r)$ intersects, is at most

$$2(128) r q^2_{n_{k+1}-2} + 2 \leq 256 r q^2_{n_{k+1}-2}.$$

Thus

$$\begin{aligned}
\mu(B(x,r)) &\leq \min\left\{\mu(J_{n_{k+1}-2}), 256 r q^2_{n_{k+1}-2} \mu(J_{n_{k+1}-1})\right\} \\
&\asymp \min\left\{\mu(J_{n_{k+1}-2}), c_1 r q^2_{n_{k+1}-2} \mu(J_{n_{k+1}-2})\right\} \\
&= \mu(J_{n_{k+1}-2}) \min\left\{1, 256 r q^2_{n_{k+1}-1}\right\} \\
&\leq c \left(\frac{1}{q^{2+\tau}_{n_{k+1}-1}}\right)^{S-\frac{10}{L}} \min\left\{1, 256 r q^2_{n_{k+1}-1}\right\} \\
&\leq c \left(\frac{1}{q^2_{n_{k+1}+1}}\right)^{S-\frac{10}{L}} \min\left\{1, 256 r q^2_{n_{k+1}+1}\right\} \\
&\leq C r^{S-\frac{10}{L}}, \text{ where } \ C = c 256^{S-\frac{10}{L}}.
\end{aligned}$$

Case III. When $n_k \leq n \leq n_{k+1} - 3$. In such a range for n, we know that $1 \leq a_n \leq M$ and $|J_n| \asymp 1/q_n^2$. So,

$$\begin{aligned}
\mu(B(x,r)) &\leq \mu(J_n) \leq c|J_n|^{S-\frac{10}{L}} \\
&\leq c \left(\frac{1}{q_n^2}\right)^{S-\frac{10}{L}} \leq c4M^2 \left(\frac{1}{q^2_{n+1}}\right)^{S-\frac{10}{L}} \\
&\ll M^2 |J_{n+1}|^{S-\frac{10}{L}} \\
&\leq M^3 g_{n+1}^{S-\frac{10}{L}} \\
&\leq M^3 r^{S-\frac{10}{L}}.
\end{aligned}$$

Conclusion

Finally, by combining all of the above cases with the mass distribution principle (Proposition 2.1.4), we have proved that

$$\dim_{\mathrm{H}} \mathcal{E}_M \geq S - 10/L.$$

Letting $L \to \infty$, we conclude that

$$\dim_{\mathrm{H}} E \geq \dim_{\mathrm{H}} \mathcal{E}_M \geq S.$$

4.4 Final remarks: the general case

The case for the general approximating function Ψ follows almost exactly the same line of investigations as for the case $\Psi(q_n) = q_n^\tau$ for any $\tau > 0$. There are some added subtleties which we will outline and then direct the reader to mimic the proof for the particular approximating function, q_n^τ, earlier.

Consider a rapidly increasing sequence $\{Q_n\}_{n\geq 1}$ of positive integers. For a fixed $\epsilon > 0$, let $\delta \geq 3\epsilon$. Define the approximating function Ψ to be

$$Q_n^{\tau-\epsilon} \leq \Psi(Q_n) \leq Q_n^{\tau+\epsilon} \text{ for all } n \geq 1,$$

where

$$\tau = \liminf_{n\to\infty} \frac{\log \Psi(Q_n)}{\log(Q_n)}.$$

Let

$$A_M = \{x \in [0,1) : 1 \leq a_n(x) \leq M, \text{ for all } n \geq 1\}.$$

For all $x \in A_M$, there exists a large $n_1 \in \mathbb{N}$ such that

$$q_{n_1-2} \leq Q_1^{1-\delta} \Longrightarrow q_{n_1-2} \leq Q_1^{1-\delta} \leq 2Mq_{n_1-2}.$$

Let

$$a_{n_1-1}(x) = \frac{1}{4}Q_1^\delta \text{ and } \frac{1}{2}q_{n_1-1}^{\tau-\epsilon} \leq a_{n_1}(x) \leq q_{n_1-1}^{\tau-\epsilon}.$$

Then the basic cylinders of order $n_1 - 2, n_1 - 1$ and n_1 can be defined as,

$$I_{n_1-2}(a_1, \ldots, a_{n_1-2}) : x \in A_M,$$

$$I_{n_1-1}\left(a_1, \ldots, a_{n_1-2}, \frac{1}{4}Q_1^\delta\right) : x \in A_M,$$

$$I_{n_1}\left(a_1, \ldots, a_{n_1-2}, \frac{1}{4}Q_1^\delta, a_{n_1}\right) : x \in A_M \text{ and } \frac{1}{2}q_{n_1-1}^{\tau-\epsilon} \leq a_{n_1}(x) \leq q_{n_1-1}^{\tau-\epsilon}.$$

Now fix the basic cylinder $I_{n_1}(a_1, \cdots, a_{n_1})$, that is, choose it to be an element in the first level of the Cantor set. Consider the set of points

$$\{[a_1, \cdots, a_{n_1}, b_1, b_2, \cdots], 1 \leq b_i \leq M \text{ for all } i \geq 1\}.$$

Then do the same as for the definition of n_1. That is, for each x, find n_2 such that q_{n_2-2} is almost Q_2.

Continuing in this way, define n_k recursively as follows. Collect the $n_k \in \mathbb{N}$ satisfying

$$q_{n_k-2} \leq Q_k^{1-\delta} \leq 2Mq_{n_k-2}.$$

Define the subset $\mathcal{E}_M^*$ of $G(\Psi) \setminus \mathcal{K}(\Psi)$ as

$$\mathcal{E}_M^* = \left\{ x \in [0,1) : \begin{array}{c} \frac{1}{2} q_{n_k-1}^{\tau-\epsilon} \le a_{n_k}(x) \le q_{n_k-1}^{\tau-\epsilon} \text{ and } a_{n_k-1}(x) = \frac{1}{4} Q_k^{\delta} \\ \text{and } 1 \le a_j(x) \le M, \text{ for all } j \ne n_k - 1, n_k \end{array} \right\}.$$

For any $n \ge 1$, define strings $(a_1, \ldots, a_n)$ by

$$D_n^* = \left\{ (a_1, \ldots, a_n) \in \mathbb{N}^n : \begin{array}{c} \frac{1}{2} q_{n_k-1}^{\tau-\epsilon} \le a_{n_k}(x) \le q_{n_k-1}^{\tau-\epsilon} \text{ and } a_{n_k-1}(x) = \frac{1}{4} Q_k^{\delta} \text{ and} \\ 1 \le a_j(x) \le M, \text{ for all } 1 \le j \ne n_k - 1, n_k \le n \end{array} \right\}.$$

For any $n \ge 1$ and $(a_1, \ldots, a_n) \in D_n^*$, define

$$J_n(a_1, \ldots, a_n) := \bigcup_{a_{n+1}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}) \tag{4.12}$$

to be the fundamental cylinder of order n, where the union in (4.12) is taken over all a_{n+1} such that $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}^*$. Then

$$\mathcal{E}_M^* = \bigcap_{n=1}^{\infty} \bigcup_{(a_1,\ldots,a_n) \in D_n^*} J_n(a_1, \ldots, a_n).$$

As can be seen, the Cantor type structure of the set $\mathcal{E}_M^*$, for the general approximating function $\Psi(Q_n)$, includes similar steps as for particular function, $\Psi(q_n) = q_n^{\tau}$, from the earlier sections. Also, the process of finding the dimension for this set follows similar steps and calculations to those for finding the dimension of the Cantor set $\mathcal{E}_M$. However, the calculations involve lengthy expressions and complicated constants. In order to avoid unnecessary intricacy, we will not produce these expressions.

4.5 Natural Question

Notice that in the set $G(\Psi) \setminus \mathcal{K}(\Psi)$ the growth rate depends on the denominator of the nth convergent 'q_n'. But if we consider the approximating function to be just a function of the index n, i.e.,

$$\mathcal{F}(\Phi) = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x) a_n(x) \ge \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}$$

where $\Phi : \mathbb{N} \to (1, \infty)$ is any function with $\lim_{n\to\infty} \Phi(n) = \infty$, then we have the following natural question.

Question 4.5.1 How big is the set $\mathcal{F}(\Phi)$ in terms of Hausdorff dimension? ■

The reason for selecting the function Φ here is just to differentiate it from function Ψ which depends on (3.1). Answering this question is more challenging as it embarks on the theory of pressure functions in determining the Hausdorff dimension. We aim to answer this question in the next chapter.

Chapter 5

A gap result in the metric theory of continued fractions

In this chapter we aim to answer Question 4.5.1.

The metrical theory of continued fractions which focuses on investigating the properties of partial quotients for almost all $x \in [0, 1)$ is one of the important areas of research in the study of continued fractions and is closely connected with the Diophantine approximation. As discussed in Chapter 1, the main connection is that the convergents of a real number x are good rational approximates for x. In fact by using Legendre's Theorem and (2.6), the set of τ-well approximable numbers for any $\tau > 1$, can be rewritten in the following form,

$$\left\{x \in [0, 1) : a_n(x) \geq q_n^{\tau-1}(x) \quad \text{for infinitely many } n \in \mathbb{N}\right\}, \tag{5.1}$$

which can be easily computed from elementary properties of continued fractions. For further details about this connection, we refer to [21]. Thus, a real number x is τ-well approximable if the partial quotients in its continued fraction expansion are growing fast. Therefore the growth rate of the partial quotients reveals how well a real number can be approximated by rationals.

Borel–Bernstein's Theorem [7, 9] which gives an analogue of Borel–Cantelli 'zero-one' law with respect to Lebesgue measure for the set of real numbers with large partial quotients, has significant role in the metrical theory of continued fractions. A lot of work has been done in the direction of improving the Borel–Bernstein's Theorem, for example, estimation of the Hausdorff dimension of sets when the partial quotient $a_n(x)$ obeys different conditions has been studied in [20, 21, 39].

Throughout this chapter, we will consider $\Phi : \mathbb{N} \to (1, \infty)$ to be an arbitrary function such that $\lim_{n\to\infty} \Phi(n) = \infty$.

Next consider the following set,

$$\mathcal{E}_1(\Phi) := \{x \in [0, 1) : a_n(x) \geq \Phi(n) \quad \text{for infinitely many } n \in \mathbb{N}\}.$$

Theorem 5.0.1 (Borel–Bernstein, [9]) *The Lebesgue measure of $\mathcal{E}_1(\Phi)$ is either zero or full according as the series $\sum_{n=1}^{\infty}\frac{1}{\Phi(n)}$ converges or diverges respectively.*

The Borel–Bernstein's Theorem is a remarkably simple dichotomy result but it fails to distinguish between exceptional sets, that is, it gives Lebesgue measure zero for sets $\mathcal{E}_1(\Phi)$ for rapidly increasing functions Φ. Recall that to distinguish between sets having Lebesgue measure zero, the notion of Hausdorff measure and dimension are the appropriate tools. Keeping this in view, Wang–Wu [45] completely determined the Hausdorff dimension of the set $\mathcal{E}_1(\Phi)$.

Theorem 5.0.2 (Wang–Wu, [45]) *Let $\Phi : \mathbb{N} \to \mathbb{R}^+$ be an arbitrary positive function. Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \text{ and } \log b = \liminf_{n\to\infty} \frac{\log\log \Phi(n)}{n}.$$

Then

$$\dim_{\mathrm{H}} \mathcal{E}_1(\Phi) = \begin{cases} s_B := \inf\{s \geq 0 : \mathsf{P}\left(T, -s\log B - s\log|T'|\right) \leq 0\} & \text{if } 1 < B < \infty; \\ \frac{1}{1+b} & \text{if } B = \infty; \\ 1 & \text{if } B = 1, \end{cases}$$

where T is the Gauss map, T' denotes the derivative of T and P represents the pressure function defined in Section 2.2. ∎

The following result illustrates the continuity of the dimensional number s_B and shows that its limit exists.

Proposition 5.0.3 (Wang–Wu, [45]) *The parameter s_B is continuous with respect to B, and*

$$\lim_{B\to 1} s_B = 1, \lim_{B\to\infty} s_B = \frac{1}{2}.$$ ∎

The set $\mathcal{E}_1(\Phi)$ is connected with the set (5.1) in the sense that in (5.1) the approximating function depends on '$q_n(x)$' whereas in $\mathcal{E}_1(\Phi)$ the approximating function Φ is a function of index 'n.' Note that the set $\mathcal{E}_1(\Phi)$ consists of those real numbers such that one partial quotient in their continued fraction expansion grows very fast but as we move towards the product of two consecutive partial quotients, the corresponding set of real numbers is linked with the set of Dirichlet non-improvable numbers (as observed by Kleinbock–Wadleigh [34]).

Recall from Chapter 3, Kleinbock–Wadleigh proved a zero-one law for the Lebesgue measure of $D(\psi)^c$. With a change of notation and Φ as defined above, we consider the set

$$\mathcal{E}_2(\Phi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Theorem 5.0.4 (Kleinbock–Wadleigh, [34]) *The Lebesgue measure of $\mathcal{E}_2(\Phi)$ is either zero or full according as the series $\sum_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)}$ converges or diverges respectively.* ∎

5.1 Statement of the main result

Note that the $\mathcal{E}_1(\Phi)$ is properly contained in $\mathcal{E}_2(\Phi)$. Since the inclusion is proper, this raises a natural question of the size of the set $\mathcal{F}(\Phi) := \mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi)$. In other words, a natural question is to estimate the size of the set

$$\mathcal{F}(\Phi) = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\},$$

in terms of Hausdorff dimension.

We prove that the set $\mathcal{F}(\Phi)$ is quite big in a sense that it is uncountable by proving that its Hausdorff dimension is positive.

Theorem 5.1.1 *Let $\Phi : \mathbb{N} \to (1,\infty)$ be any function with $\lim_{n\to\infty} \Phi(n) = \infty$. Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \quad \textit{and} \quad \log b = \liminf_{n\to\infty} \frac{\log\log \Phi(n)}{n}. \tag{5.2}$$

Then

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) = \begin{cases} t_B := \inf\{s \geq 0 : \mathsf{P}\left(T, -s^2 \log B - s \log |T'|\right) \leq 0\} & \textit{if } 1 < B < \infty; \\ \frac{1}{1+b} & \textit{if } B = \infty, \end{cases}$$ ∎

where T is the Gauss map and P represents the pressure function defined in Section 2.2.

Note that if we take $B = 1$ then from the definition of $\mathcal{F}(\Phi)$ we have $a_{n+1}(x) < 1$ which is a contradiction to the assumption that $a_{n+1}(x) \geq 1$. Therefore, B is strictly greater than 1.

Throughout this chapter we consider the specific potential, that is,

$$\varphi_1(x) = -s^2 \log B - s \log |T'(x)|,$$

where $1 < B < \infty$, $s \geq 0$ and T' is the derivative of Gauss map T. By applying Proposition 2.2.2 to φ_1, it is easy to check that φ_1 satisfies the variation condition.

Therefore, by considering the value of potential φ_1 in definition of pressure function (2.10), we have

$$\begin{aligned}\mathsf{P}_{\mathcal{A}}(T, -s^2 \log B - s \log |T'|) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\ldots,a_n \in \mathcal{A}} e^{S_n \varphi_1(x)} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\ldots,a_n \in \mathcal{A}} \left(\frac{1}{B^{ns} q_n^2} \right)^s.\end{aligned}$$

Let $n \geq 1$. For the requirement of this chapter, define

$$\begin{aligned}t_{n,B}(\mathcal{A}) &:= \inf \left\{ s \geq 0 : \sum_{a_1,\ldots,a_n \in \mathcal{A}} \frac{1}{(B^{ns} q_n^2)^s} \leq 1 \right\}; \\ t_B(\mathcal{A}) &:= \inf\{s \geq 0 : \mathsf{P}_{\mathcal{A}}(T, -s^2 \log B - s \log |T'|) \leq 0\}; \\ t_B(\mathbb{N}) &:= \inf\{s \geq 0 : \mathsf{P}(T, -s^2 \log B - s \log |T'|) \leq 0\}.\end{aligned}$$

For any $M \in \mathbb{N}$, take $\mathcal{A}_M = \{1, 2, \ldots, M\}$. For simplicity, write $t_{n,B}(M)$ for $t_{n,B}(\mathcal{A}_M)$, $t_B(M)$ for $t_B(\mathcal{A}_M)$, $t_{n,B}$ for $t_{n,B}(\mathbb{N})$ and t_B for $t_B(\mathbb{N})$. From Proposition 2.2.2 and by the definition of $t_{n,B}(M)$, we have

$$\lim_{n\to\infty} t_{n,B}(M) = t_B(M), \quad \lim_{M\to\infty} t_B(M) = t_B. \tag{5.3}$$

Since B belongs to $(1, \infty)$, therefore the dimensional number t_B is continuous with respect to B and

$$\lim_{B\to 1} t_B = 1, \quad \lim_{B\to\infty} t_B = \frac{1}{2}.$$

Also, from (2.2) and the definition of $t_{n,B}(M)$, we have $0 \leq t_B(M) \leq 1$.

5.2 Proof of Theorem 5.1.1

Proof: The proof of Theorem 5.1.1 consist of two cases.

Case 1: $1 < B < \infty$.

By supposition (5.2) in the statement of Theorem 5.1.1, one can easily note that

$$\dim_H \mathcal{F}(\Phi) = \dim_{\mathrm{H}} \mathcal{F}(\Phi : n \to B^n) \quad \text{when } 1 < B < \infty.$$

Therefore, we can simply take the approximating function $\Phi(n) := B^n$ and rewrite the set $\mathcal{F}(\Phi)$ as

$$\mathcal{F}(B) := \left\{ x \in [0,1) : \begin{array}{r} a_n(x) a_{n+1}(x) \geq B^n \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < B^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

The aim is to show $\dim_H \mathcal{F}(B) = t_B$. The details of the proof of Theorem 5.1.1 are divided into two further subsections. That is finding the upper bound $\dim_{\mathrm{H}} \mathcal{F}(B) \leq t_B$; and the lower bound $\dim_{\mathrm{H}} \mathcal{F}(B) \geq t_B$, separately. Taken together, this will conclude our proof for **Case 1**.

The upper bound for $\mathcal{F}(B)$

For the upper bound of $\dim_{\mathrm{H}} \mathcal{F}(B)$, we consider two sets:

$$\mathcal{F}_1(B) = \Big\{x \in [0,1) : a_n(x) \geq B^n \ \text{ for infinitely many } n \in \mathbb{N}\Big\} \text{ and}$$

$$\mathcal{F}_2(B) = \left\{x \in [0,1) : \begin{array}{l} 1 \leq a_n(x) \leq B^n, a_{n+1}(x) \geq B^n/a_n(x) \text{ for infinitely many} \\ n \in \mathbb{N} \text{ and } a_{n+1}(x) < B^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array}\right\}.$$

From the definition of Hausdorff dimension it follows that

$$\dim_{\mathrm{H}} \mathcal{F}(B) \leq \max\{\dim_{\mathrm{H}} \mathcal{F}_1(B), \dim_{\mathrm{H}} \mathcal{F}_2(B)\}.$$

The Hausdorff dimension of $\mathcal{F}_1(B)$ follows from Theorem 5.0.2. So it remains to obtain the upper bound for the Hausdorff dimension of $\mathcal{F}_2(B)$. Recall that the pressure function $P(T, .)$ is monotonic with respect to the potential which implies then $s_B \leq t_B$. So, once we can show $\dim_{\mathrm{H}} \mathcal{F}_2(B) \leq t_B$, the upper bound for $\dim_{\mathrm{H}} \mathcal{F}(B)$ follows.

Fix $\epsilon > 0$ and let $s = t_B + 2\epsilon$. We will show that $\dim_{\mathrm{H}} \mathcal{F}_2(B) \leq s$.

By the definition of t_B, one has for any n large,

$$\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{ns}q_{n-1}^2}\right)^s \leq \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{n(t_B+\epsilon)}q_{n-1}^2}\right)^{t_B+\epsilon} \cdot B^{-n\epsilon^2} \leq B^{-n\epsilon^2}. \tag{5.4}$$

Recall that

$$\begin{aligned}
\mathcal{F}_2(B) &= \left\{x \in [0,1) : \begin{array}{l} 1 \leq a_n(x) \leq B^n, a_{n+1}(x) \geq B^n/a_n(x) \text{ for infinitely many} \\ n \in \mathbb{N} \text{ and } a_{n+1}(x) < B^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array}\right\} \\
&\subset \left\{x \in [0,1) : \begin{array}{l} 1 \leq a_n(x) \leq B^n, (B^n/a_n(x)) \leq a_{n+1}(x) < B^n \\ \text{for infinitely many } n \in \mathbb{N} \end{array}\right\} \\
&= \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \left\{x \in [0,1) : \begin{array}{l} 1 \leq a_n(x) \leq B^n, \\ (B^n/a_n(x)) \leq a_{n+1}(x) < B^n \end{array}\right\} \\
&= \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \mathcal{F}_I \cup \mathcal{F}_{II}
\end{aligned} \tag{5.5}$$

where

$$\mathcal{F}_I = \{x \in [0,1) : 1 \leq a_n(x) < \alpha^n, (B^n/a_n(x)) \leq a_{n+1}(x) < B^n\}$$
$$\mathcal{F}_{II} = \{x \in [0,1) : \alpha^n \leq a_n(x) \leq B^n, (B^n/a_n(x)) \leq a_{n+1}(x) < B^n\}$$

and $\alpha^n > 1$. Here we have assumed that $\alpha \in \mathbb{R}$ with $\alpha > 1$ and therefore $\alpha^n > 1$, for all $n \in \mathbb{N}$.

Next we will separately find suitable coverings for sets $\mathcal{F}_I$ and $\mathcal{F}_{II}$. Then the union of the coverings for both these sets will serve as an appropriate covering for $\mathcal{F}_2(B)$. To proceed, assume that for some some s, we have $\alpha = B^s$.

The set $\mathcal{F}_I$ can be covered by collections of fundamental cylinders J_n of order n:

$$\begin{aligned}
\mathcal{F}_I &\subset \{x \in [0,1) : 1 \le a_n(x) \le \alpha^n, (B^n/a_n(x)) \le a_{n+1}(x)\} \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left\{ x \in [0,1) : \begin{array}{l} a_k(x) = a_k, 1 \le k \le n-1, 1 \le a_n(x) \le \alpha^n, \\ (B^n/a_n(x)) \le a_{n+1}(x) \end{array} \right\} \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \bigcup_{1\le a_n<\alpha^n} \bigcup_{a_{n+1}\ge B^n/a_n} I_{n+1}(a_1,\cdots,a_{n+1}) \\
&= \bigcup_{\substack{a_1,\cdots,a_{n-1}\in\mathbb{N}, \\ 1\le a_n\le\alpha^n}} J_n(a_1,\cdots,a_n).
\end{aligned}$$

Note that since

$$J_n(a_1,\cdots,a_n) = \bigcup_{a_{n+1}\ge B^n/a_n} I_{n+1}(a_1,\cdots,a_{n+1}),$$

we have

$$|J_n(a_1,\cdots,a_n)| \asymp \frac{1}{B^n a_n q_{n-1}^2}.$$

Cover the set $\mathcal{F}_{II}$ by the collection of fundamental cylinders J'_{n-1} of order $n-1$:

$$\begin{aligned}
\mathcal{F}_{II} &\subset \left\{ x \in [0,1) : a_n(x) \ge \alpha^n \right\} \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left\{ x \in [0,1) : a_k(x) = a_k, 1 \le k \le n-1, a_n(x) \ge \alpha^n \right\} \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \bigcup_{a_n\ge\alpha^n} I_n(a_1,\cdots,a_n) \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} J'_{n-1}(a_1,\cdots,a_{n-1}).
\end{aligned}$$

Since

$$J'_{n-1}(a_1,\cdots,a_{n-1}) = \bigcup_{a_n\ge\alpha^n} I_n(a_1,\cdots,a_n),$$

we have

$$|J'_{n-1}(a_1,\cdots,a_{n-1})| \asymp \frac{1}{\alpha^n q_{n-1}^2}.$$

Now we consider the s-volume of the cover of $\mathcal{F}_I \bigcup \mathcal{F}_{II}$:

$$\begin{aligned}
&\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \sum_{1\le a_n\le\alpha^n} \left(\frac{1}{B^n a_n q_{n-1}^2}\right)^s + \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s \\
\asymp &\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \alpha^{n(1-s)} \left(\frac{1}{B^n q_{n-1}^2}\right)^s + \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s \quad \text{(integrating on } a_n\text{)} \\
= &\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left[\left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s + \left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s \right] \quad \text{(by } \alpha = B^s\text{)}
\end{aligned}$$

$$\asymp \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{ns}q_{n-1}^2}\right)^s.$$

Therefore, from equation (5.5), we obtain

$$\mathcal{F}_2(B) \subset \bigcap_{N=1}^{\infty}\bigcup_{n\geq N}\left\{\bigcup_{\substack{a_1,\cdots,a_{n-1}\in\mathbb{N}\\ 1\leq a_n\leq\alpha^n}} J_n(a_1,\cdots,a_n) \cup \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} J'_{n-1}(a_1,\cdots,a_{n-1})\right\}. \tag{5.6}$$

Thus from equations (5.6) and (5.4), we obtain the s-dimensional Hausdorff measure of $\mathcal{F}_2(B)$ as

$$\mathcal{H}^s(\mathcal{F}_2(B)) \leq \liminf_{N\to\infty}\sum_{n\geq N}^{\infty}\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}}\left(\frac{1}{B^{ns}q_{n-1}^2}\right)^s \leq \liminf_{N\to\infty}\sum_{n\geq N}^{\infty}\frac{1}{B^{n\epsilon^2}} = 0.$$

This gives $\dim_{\mathrm{H}}\mathcal{F}_2(B) \leq s = t_B + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\dim_{\mathrm{H}}\mathcal{F}_2(B) \leq t_B$. Consequently,

$$\dim_{\mathrm{H}}\mathcal{F}(B) \leq t_B. \tag{5.7}$$

The lower bound for $\mathcal{F}(B)$

In this subsection we will determine the lower bound for $\dim_{\mathrm{H}}\mathcal{F}(B)$. Here the pressure function material will be utilised.

To prove $\dim_{\mathrm{H}}\mathcal{F}(B) \geq t_B$, from (5.3) it is sufficient to show that $\dim_{\mathrm{H}}\mathcal{F}(B) \geq t_{L,B}(M)$ for all large enough M and L. To proceed we will construct a subset $\mathcal{F}_M(B) \subset \mathcal{F}(B)$ and use the lower bound for the Hausdorff dimension of $\mathcal{F}_M(B)$ to approximate that of $\mathcal{F}(B)$.

Fix $s < t_{L,B}(M)$. Let $\alpha = B^s$ where $\alpha \leq B$ and $\alpha > 1$. Choose a rapidly increasing sequence of integers $\{n_k\}_{k\geq 1}$ and for convenience, we let $n_0 = 0$.

Define the subset $\mathcal{F}_M(B)$ of $\mathcal{F}(B)$ as follows

$$\mathcal{F}_M(B) = \left\{x\in[0,1) : \begin{array}{l}\frac{B^{n_k}}{2\alpha^{n_k}} \leq a_{n_k+1}(x) \leq \frac{B^{n_k}}{\alpha^{n_k}}, a_{n_k}(x) = 2\alpha^{n_k} \text{ for all } k\geq 1\\ \text{and } 1\leq a_j(x)\leq M, \text{ for all } j\neq n_k, n_k+1\end{array}\right\}. \tag{5.8}$$

Structure of $\mathcal{F}_M(B)$

For any $n \geq 1$, define the set of strings

$$D_n = \left\{(a_1,\ldots,a_n)\in\mathbb{N}^n : \begin{array}{l}\frac{B^{n_k}}{2\alpha^{n_k}} \leq a_{n_k+1}(x) \leq \frac{B^{n_k}}{\alpha^{n_k}}, a_{n_k}(x) = 2\alpha^{n_k}\\ \text{and } 1\leq a_j(x)\leq M,\ j\neq n_k, n_k+1\end{array}\right\}.$$

Recall that for any $n \geq 1$ and $(a_1, \ldots, a_n) \in D_n$, we call $I_n(a_1, \ldots, a_n)$ a basic cylinder of order n and

$$J_n := J_n(a_1, \ldots, a_n) := \bigcup_{a_{n+1}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}) \tag{5.9}$$

a fundamental cylinder of order n, where the union in (5.9) is taken over all a_{n+1} such that $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}$.

Note that in (5.8) according to the limitations on the partial quotients we have three distinct cases for J_n. For $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}$:

$$n_{k-1} + 1 \leq n \leq n_k - 2, \qquad J_n = \bigcup_{1 \leq a_{n+1} \leq M} I_{n+1}(a_1, \ldots, a_n, a_{n+1}), \tag{5.10}$$

$$n = n_k - 1, \qquad J_n = \bigcup_{a_{n+1} = 2\alpha^n} I_{n+1}(a_1, \ldots, a_n, a_{n_1+1}), \tag{5.11}$$

$$n = n_k, \qquad J_n = \bigcup_{\frac{B^n}{2\alpha^n} \leq a_{n+1} \leq \frac{B^n}{\alpha^n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}). \tag{5.12}$$

Then,

$$\mathcal{F}_M(B) = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \ldots, a_n) \in D_n} J_n(a_1, \ldots, a_n).$$

Lengths of fundamental cylinders

In the following subsection we will estimate the lengths of the fundamental cylinders defined in last subsection.

I. If $n_{k-1} + 1 \leq n \leq n_k - 2$ then from (5.10) and using (2.2), we have

$$\begin{aligned}
|J_n(a_1, \ldots, a_n)| &= \sum_{1 \leq a_{n+1} \leq M} |I_{n+1}(a_1, \ldots, a_n, a_{n+1})| \\
&= \sum_{1 \leq a_{n+1} \leq M} \frac{1}{q_{n+1}(q_{n+1} + q_n)} \qquad (5.13) \\
&= \sum_{a_{n+1}=1}^{M} \frac{1}{q_n} \left(\frac{1}{q_{n+1}} - \frac{1}{q_{n+1} + q_n} \right) \\
&= \frac{1}{q_n} \sum_{a_{n+1}=1}^{M} \left(\frac{1}{a_{n+1} q_n + q_{n-1}} - \frac{1}{(a_{n+1} + 1) q_n + q_{n-1}} \right) \\
&= \frac{1}{q_n} \left(\frac{1}{q_n + q_{n-1}} - \frac{1}{(M+1) q_n + q_{n-1}} \right) \\
&= \frac{M}{((M+1) q_n + q_{n-1})(q_n + q_{n-1})}. \qquad (5.14)
\end{aligned}$$

Also, from (5.13), we have

$$\frac{1}{6q_n^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{1}{q_n^2}. \tag{5.15}$$

In particular for $n = n_k + 1$,

$$\frac{1}{24B^{2n}q_{n-2}^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{1}{4B^{2n}q_{n-2}^2}. \tag{5.16}$$

II. If $n = n_k - 1$ then from (5.11) and following the same steps as for **I** ,we have

$$|J_n(a_1, \ldots, a_n)| = \frac{1}{(2\alpha^n q_n + q_{n-1})((2\alpha^n + 1)q_n + q_{n-1})} \tag{5.17}$$

and

$$\frac{1}{12\alpha^{n+1}q_n^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{1}{2\alpha^{n+1}q_n^2}. \tag{5.18}$$

III. If $n = n_k$ then from (5.12) and following the similar steps as for **I**, we obtain

$$|J_n(a_1, \ldots, a_n)| = \frac{\frac{B^n}{2\alpha^n} + 1}{(\frac{B^n}{2\alpha^n} q_n + q_{n-1})((\frac{B^n}{\alpha^n} + 1)q_n + q_{n-1})} \tag{5.19}$$

and

$$\frac{\alpha^n}{6B^n q_n^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{2\alpha^n}{B^n q_n^2}.$$

Further,

$$\frac{1}{32\alpha^n B^n q_{n-1}^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{1}{2\alpha^n B^n q_{n-1}^2}. \tag{5.20}$$

Supporting measure on $\mathcal{F}_M(B)$

To construct a suitable measure supported on $\mathcal{F}_M(B)$ first recall that $t_{L,B}(M)$ is the solution to

$$\sum_{a_1,\ldots,a_L \in \mathcal{A}_M} \left(\frac{1}{B^{Ls}q_L^2}\right)^s = 1.$$

For $\alpha = B^s$ this sum becomes

$$\sum_{a_1,\ldots,a_L \in \mathcal{A}_M} \left(\frac{1}{\alpha^L q_L^2}\right)^s = 1.$$

Let $m_k L = n_k - n_{k-1} - 1$ for any $k \geq 1$. Note that $m_1 L = n_1 - 1$ since we have assumed $n_0 = 0$. Define

$$w = \sum_{a_1,\ldots,a_L \in \mathcal{A}_M} \left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+t+1}, \cdots, a_{n_{k-1}+(t+1)L})}\right)^s$$

where $0 \leq t \leq m_k - 1$.

Step I. Let $1 \leq m \leq m_1$. We first define a positive measure for the fundamental cylinder $J_{mL}(a_1, \ldots, a_{mL})$ as

$$\mu(J_{mL}(a_1, \ldots, a_{mL})) = \prod_{t=0}^{m-1} \frac{1}{w} \left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})}\right)^s,$$

and then we distribute this measure uniformly over its next offspring.
Step II. When $n = m_1L = n_1 - 1$ then define a measure

$$\mu(J_{m_1L}(a_1, \ldots, a_{m_1L})) = \prod_{t=0}^{m_1-1} \frac{1}{w} \left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})} \right)^s.$$

Step III. When $n = m_1L + 1 = n_1$ then for $J_{n_1}(a_1, \ldots, a_{n_1})$, define a measure

$$\mu(J_{n_1}(a_1, \ldots, a_{n_1})) = \frac{1}{2\alpha^{n_1}} \mu(J_{n_1-1}(a_1, \ldots, a_{n_1-1})).$$

In other words, the measure of J_{n_1-1} is uniformly distributed on its next offspring J_{n_1}.
Step IV. When $n = n_1 + 1$, define

$$\mu(J_{n_1+1}(a_1, \ldots, a_{n_1+1})) = \frac{2\alpha^{n_1}}{B^{n_1}} \mu(J_{n_1}(a_1, \ldots, a_{n_1})).$$

The measure of other fundamental cylinders of level less than $n_1 - 1$ is given by the consistency of a measure. To be more precise, for any $n < n_1 - 1$, suppose

$$\mu(J_n(a_1, \cdots, a_n)) = \sum_{J_{m_1L} \subset J_n} \mu(J_{m_1L}).$$

So for any $m < m_1$, the measure of fundamental cylinder J_{mL} is given by

$$\begin{aligned}
\mu(J_{mL}(a_{n_{k-1}+t+1}, \cdots, a_{n_{k-1}+(t+1)L})) &= \sum_{J_{m_1L} \subset J_{mL}} \mu(J_{m_1}L) \\
&= \prod_{t=0}^{m-1} \frac{1}{w} \left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})} \right)^s.
\end{aligned}$$

The measure of fundamental cylinders for other levels can be defined inductively.

For $k \geq 2$ define,

$$\begin{aligned}
\mu(J_{n_k-1}(a_1, \ldots, a_{n_k-1})) &= \mu\left(J_{n_{k-1}+1}(a_1, \ldots, a_{n_{k-1}+1})\right) \\
&\quad \cdot \prod_{t=0}^{m_k-1} \frac{1}{w} \left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1}, \ldots, a_{n_{k-1}+(t+1)L})} \right)^s, \\
\mu(J_{n_k}(a_1, \ldots, a_{n_k})) &= \frac{1}{2\alpha^{n_k}} \mu\left(J_{n_k-1}(a_1, \ldots, a_{n_k-1})\right) \text{ and} \\
\mu(J_{n_k+1}(a_1, \ldots, a_{n_k+1})) &= \frac{2\alpha^{n_k}}{B^{n_k}} \mu\left(J_{n_k}(a_1, \ldots, a_{n_k})\right) \cdot
\end{aligned}$$

The Hölder exponent of the measure μ

Similar to last chapter, first we estimate the Hölder exponent of $\mu\left(J_n(a_1, \ldots, a_n)\right)$ in relation to $|J_n(a_1, \ldots, a_n)|$ and then Hölder condition between the ball $\mu(B(x, r))$ and radius r.
Estimation of $\mu(J_n(a_1, \ldots, a_n))$.

In this subsection we will estimate the measure μ of the fundamental cylinders for several cases defined in Section 5. For this we split the process into several cases. Recall that $\alpha > 1$ which implies $\alpha^L > 1$, for any L large enough. For sufficiently large k_0 choose $\epsilon_0 > \frac{n_{k-1}}{n_k} + \frac{1}{n_k}$ such that

$$\frac{m_k L}{n_k} = 1 - \frac{n_{k-1}}{n_k} - \frac{1}{n_k} \geq 1 - \epsilon_0, \text{ for all } k > k_0. \tag{5.21}$$

Case I. $n = mL$ for some $1 \leq m < m_1$.

$$\begin{aligned}
\mu(J_{mL}(a_1, \dots, a_{mL})) &\leq \prod_{t=0}^{m-1} \left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \dots, a_{(t+1)L})} \right)^s \\
&\overset{(2.5)}{\leq} 4^{m-1} \cdot \left(\frac{1}{q_{mL}^2(a_1, \dots, a_{mL})} \right)^s \\
&\overset{(2.3)}{\leq} \left(\frac{1}{q_{mL}^2(a_1, \dots, a_{mL})} \right)^{s-\frac{2}{L}} \\
&\overset{(5.15)}{\leq} 6|J_{mL}(a_1, \dots, a_{mL})|^{s-\frac{2}{L}}.
\end{aligned}$$

Case 2. $n = m_1 L = n_1 - 1$.

$$\begin{aligned}
\mu(J_{m_1 L}(a_1, \dots, a_{m_1 L})) &\leq \prod_{t=0}^{m_1-1} \left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \dots, a_{(t+1)L})} \right)^s \\
&\leq \left(\frac{1}{\alpha^{m_1 L}} \right)^s \left(\frac{1}{q_{m_1 L}^2(a_1, \dots, a_{m_1 L})} \right)^{s-\frac{2}{L}} \\
&\overset{(5.21)}{\leq} \left(\frac{1}{\alpha^{1-\epsilon_0}} \right)^{s n_1} \left(\frac{1}{q_{m_1 L}^2(a_1, \dots, a_{m_1 L})} \right)^{s-\frac{2}{L}} \\
&\leq \left(\frac{1}{\alpha^{n_1} q_{n_1-1}^2} \right)^{s-\frac{2}{L}-\epsilon_0} \qquad (5.22) \\
&\overset{(5.18)}{\leq} 12|J_{m_1 L}(a_1, \dots, a_{m_1 L})|^{s-\frac{2}{L}-\epsilon_0}.
\end{aligned}$$

Case 3. $n = m_1 L + 1 = n_1$.

$$\begin{aligned}
\mu(J_{n_1}(a_1, \dots, a_{n_1})) &= \frac{1}{2\alpha^{n_1}} \mu(J_{n_1-1}(a_1, \dots, a_{n_1-1})) \\
&\overset{(5.22)}{\leq} \frac{1}{2\alpha^{n_1}} \left(\frac{1}{\alpha^{n_1} q_{n_1-1}^2} \right)^{s-\frac{2}{L}-\epsilon_0} \\
&= \frac{1}{2B^{s n_1}} \left(\frac{1}{\alpha^{n_1} q_{n_1-1}^2} \right)^{s-\frac{2}{L}-\epsilon_0} \quad (\ \alpha = B^s) \\
&\leq \frac{1}{2} \left(\frac{1}{B^{n_1} \alpha^{n_1} q_{n_1-1}^2} \right)^{s-\frac{2}{L}-\epsilon_0} \\
&\overset{(5.20)}{\leq} 16|J_{n_1}(a_1, \dots, a_{n_1})|^{s-\frac{2}{L}-\epsilon_0}.
\end{aligned}$$

Case 4. $n = n_1 + 1$.

$$\begin{aligned}
\mu(J_{n_1+1}(a_1,\dots,a_{n_1+1})) &= \frac{2\alpha^{n_1}}{B^{n_1}}\mu(J_{n_1}(a_1,\dots,a_{n_1})) \\
&\le \frac{2\alpha^{n_1}}{2B^{n_1}}\left(\frac{1}{B^{n_1}\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0} \\
&\le \left(\frac{1}{B^{2n_1}\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0} \\
&\overset{(5.16)}{\le} 24|J_{n_1+1}(a_1,\dots,a_{n_1+1})|^{s-\frac{2}{L}-\epsilon_0}.
\end{aligned}$$

Here for the second inequality, we use $B/\alpha \ge (B/\alpha)^s$ which is always true for $\alpha \le B$ and $s \le 1$.

For a general fundamental cylinder, we only give the estimation on the measure of cylinder $J_{n_k-1}(a_1,\dots,a_{n_k-1})$. The estimation for other fundamental cylinders can be carried out similarly. Recall that

$$\begin{aligned}
\mu(J_{n_k-1}(a_1,\dots,a_{n_k-1})) =&\mu\left(J_{n_{k-1}+1}(a_1,\dots,a_{n_{k-1}+1})\right) \\
&\cdot\prod_{t=0}^{m_k-1}\frac{1}{w}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1},\dots,a_{n_{k-1}+(t+1)L})}\right)^s.
\end{aligned}$$

This further implies,

$$\begin{aligned}
\mu\left(J_{n_k-1}(a_1,\dots,a_{n_k-1})\right) \le& \left[\prod_{j=1}^{k-1}\left(\frac{1}{B^{n_j}}\prod_{t=0}^{m_j-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{j-1}+tL+1},\dots,a_{n_{j-1}+(t+1)L})}\right)^s\right)\right] \\
&\cdot\prod_{t=0}^{m_k-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1},\dots,a_{n_{k-1}+(t+1)L})}\right)^s \\
\le& \left[\prod_{j=1}^{k-1}\left(\frac{1}{B^{n_j}}\prod_{t=0}^{m_j-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{j-1}+tL+1},\dots,a_{n_{j-1}+(t+1)L})}\right)^s\right)\right] \\
&\cdot\prod_{t=0}^{m_k-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1},\dots,a_{n_{k-1}+(t+1)L})}\right)^s.
\end{aligned}$$

By similar arguments as used in **Case 4** for the first product and **Case 2** for the second product, we obtain

$$\begin{aligned}
\mu\left(J_{n_k-1}(a_1,\dots,a_{n_k-1})\right) \le& \prod_{j=1}^{k-1}\left(\frac{1}{B^{2n_j}q_{m_jL}^2(a_{n_{j-1}+tL+1},\dots,a_{n_{j-1}+(t+1)L})}\right)^{s-\frac{2}{L}-\epsilon_0} \\
&\cdot\left(\frac{1}{\alpha^{n_k}q_{m_kL}^2(a_{n_{k-1}+tL+1},\dots,a_{n_{k-1}+(t+1)L})}\right)^{s-\frac{2}{L}-\epsilon_0} \\
\le& 4^{2k}\left(\frac{1}{\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0} \le \left(\frac{1}{\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}
\end{aligned}$$

$$\overset{(5.18)}{\leq} 12|J_{n_k-1}(a_1,\ldots,a_{n_k-1})|^{s-\frac{6}{L}-\epsilon_0}.$$

Consequently,

$$\begin{aligned}\mu(J_{n_k}(a_1,\ldots,a_{n_k})) &= \frac{1}{2\alpha^{n_k}}\mu\left(J_{n_{k-1}}(a_1,\ldots,a_{n_{k-1}})\right)\\ &\leq \frac{1}{2B^{sn_k}}\left(\frac{1}{\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}\\ &\leq \frac{1}{2}\left(\frac{1}{B^{n_k}\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}\\ &\overset{(5.20)}{\leq} 16|J_{n_k}(a_1,\ldots,a_{n_k})|^{s-\frac{6}{L}-\epsilon_0}.\end{aligned}$$

In summary, we have shown that for any $n\geq 1$ and $(a_1,\ldots,a_n)\in D_n$, we get

$$\mu\left(J_n\left(a_1,\ldots,a_n\right)\right) \ll \left|J_n\left(a_1,\ldots,a_n\right)\right|^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}.$$

Gap estimation.

First we estimate the gaps between the adjoint fundamental cylinders of same order which will be useful for estimating $\mu(B(x,r))$.

Let us start by assuming n is even (similar steps can be followed when n is odd). Then for $(a_1,\ldots,a_n)\in D_n$, given a fundamental cylinder $J_n\left(a_1,\ldots,a_n\right)$, represent the distance between $J_n\left(a_1,\ldots,a_n\right)$ and its left (respectively right) adjoint fundamental cylinder say

$$J_n' = J_n'(a_1,\cdots,a_{n-1},a_n-1) \text{ (if it exists)}$$

(respectively, $J_n''=J_n''(a_1,\cdots,a_{n-1},a_n+1)$ if it exits) of order n by

$$g^l(a_1,\ldots,a_n)$$

(respectively, $g^r(a_1,\ldots,a_n)$). Let

$$G_n\left(a_1,\ldots,a_n\right) = \min\left\{g^r\left(a_1,\ldots,a_n\right), g^l\left(a_1,\ldots,a_n\right)\right\}.$$

We will consider three different cases according to the range of n as in (5.10)-(5.12) for $\mathcal{F}_M(B)$ in order to estimate the lengths of gaps on both sides of fundamental cylinders $J_n\left(a_1,\ldots,a_n\right)$.

Gap I. $n_{k-1}+1\leq n\leq n_k-2$, for all $k\geq 1$.

There exists a basic cylinder of order n contained in $I_{n-1}\left(a_1,\ldots,a_{n-1}\right)$ which lies on the left of $I_n\left(a_1,\ldots,a_n\right)$ and also there exists a basic cylinder of order n contained in $I_{n-1}\left(a_1,\ldots,a_{n-1}\right)$ which lies on the right of $I_n\left(a_1,\ldots,a_n\right)$. In this case, $(a_1,\ldots,a_n-1)\in D_n$ and $(a_1,\ldots,a_n+1)\in D_n$, whereas $g^l\left(a_1,\ldots,a_n\right)$ is just the

distance between the right endpoint of $J_n'(a_1,\ldots,a_n-1)$ and the left endpoint of $J_n(a_1,\ldots,a_n)$.
The right endpoint of $J_n'(a_1,\ldots,a_n-1)$ is the same as the left endpoint of $I_n(a_1,\ldots,a_n)$. Since n is even, from equation (2.1) this has formula $\frac{p_n}{q_n}$.
Note that the left endpoint of $J_n(a_1,\ldots,a_n)$ lies on the extreme left of all the constituent cylinders $\{I_{n+1}(a_1,\ldots,a_{n+1}) : 1 \le a_{n+1} \le M\}$. This tells us that $a_{n+1} = M$. Since $n+1$ is odd, again from equation (2.1) this has formula

$$\frac{(Mp_n + p_{n-1}) + p_n}{(Mq_n + q_{n-1}) + p_n} = \frac{(M+1)\,p_n + p_{n-1}}{(M+1)\,q_n + q_{n-1}}.$$

Therefore, we have

$$\begin{aligned} g^l(a_1,\ldots,a_n) &= \frac{(M+1)\,p_n + p_{n-1}}{(M+1)\,q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{p_{n-1}q_n - q_{n-1}p_n}{((M+1)\,q_n + q_{n-1})\,q_n} = \frac{1}{((M+1)\,q_n + q_{n-1})\,q_n}. \end{aligned}$$

In this case $g^r(a_1,\ldots,a_n)$ is just the distance between the right endpoint of $J_n(a_1,\ldots,a_n)$ and the left endpoint of $J_n''(a_1,\ldots,a_n+1)$.

The right endpoint of $J_n(a_1,\ldots,a_n)$ is the same as the right endpoint of $I_n(a_1,\ldots,a_n)$. Since n is even, again using equation (2.1) this has formula

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

Also, the left endpoint of $J_n''(a_1,\ldots,a_n+1)$ lies on the extreme left of all the constituent cylinders $\{I_{n+1}(a_1,\ldots,a_{n-1},a_n+1,a_{n+1}) : 1 \le a_{n+1} \le M\}$. This tells us that $a_{n+1} = M$. Since $n+1$ is odd, we have

$$\frac{(M+1)\,[(a_n+1)\,p_{n-1} + p_{n-2}] + p_{n-1}}{(M+1)\,[(a_n+1)\,q_{n-1} + q_{n-2}] + q_{n-1}} = \frac{(M+1)\,(p_n + p_{n-1}) + p_{n-1}}{(M+1)\,(q_n + q_{n-1}) + q_{n-1}}.$$

Therefore,

$$\begin{aligned} g^r(a_1,\ldots,a_n) &= \frac{(M+1)\,(p_n + p_{n-1}) + p_{n-1}}{(M+1)\,(q_n + q_{n-1}) + q_{n-1}} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \\ &= \frac{1}{((M+1)\,(q_n + q_{n-1}) + q_{n-1})\,(q_n + q_{n-1})}. \end{aligned}$$

Hence

$$G_n(a_1,\ldots,a_n) = \frac{1}{((M+1)\,(q_n + q_{n-1}) + q_{n-1})\,(q_n + q_{n-1})}$$

and from (5.14), we notice that

$$G_n(a_1,\ldots,a_n) \ge \frac{1}{2M}|J_n(a_1,\ldots,a_n)|.$$

Gap II. $n = n_k - 1$.

In this case the $g^l(a_1,\ldots,a_n)$ is larger than the distance between the left endpoint of $I_n(a_1,\ldots,a_n)$ and the left endpoint of $J_n(a_1,\ldots,a_n)$ whereas $g^r(a_1,\ldots,a_n)$ is larger than the distance between the right endpoint of $I_n(a_1,\ldots,a_n)$ and the right endpoint of $J_n(a_1,\ldots,a_n)$.
Thus proceeding in the similar way as in **Gap I**, we obtain

$$\begin{aligned} g^l(a_1,\ldots,a_n) &\geq \frac{(2\alpha^n+1)p_n+p_{n-1}}{(2\alpha^n+1)q_n+q_{n-1}}-\frac{p_n}{q_n} \\ &= \frac{1}{((2\alpha^n+1)q_n+q_{n-1})q_n}, \end{aligned}$$

and

$$\begin{aligned} g^r(a_1,\ldots,a_n) &\geq \frac{p_n+p_{n-1}}{q_n+q_{n-1}}-\frac{(2\alpha^n+1)p_n+p_{n-1}}{(2\alpha^n+1)q_n+q_{n-1}} \\ &= \frac{1}{((2\alpha^n+1)q_n+q_{n-1})(q_n+q_{n-1})}. \end{aligned}$$

Therefore,

$$G_n(a_1,\ldots,a_n) \geq \frac{1}{((2\alpha^n+1)q_n+q_{n-1})(q_n+q_{n-1})},$$

and from (5.17),

$$G_n(a_1,\ldots,a_n) \geq \frac{1}{2}|J_n(a_1,\ldots,a_n)|.$$

Gap III. $n=n_k$.

Following the similar arguments as in **Gap II**, we have

$$\begin{aligned} g^l(a_1,a_2,\ldots,a_n) &\geq \frac{\left(\frac{B^n}{\alpha^n}+1\right)p_n+p_{n-1}}{\left(\frac{B^n}{\alpha^n}+1\right)q_n+q_{n-1}}-\frac{p_n}{q_n} \\ &= \frac{1}{\left(\left(\frac{B^n}{\alpha^n}+1\right)q_n+q_{n-1}\right)q_n}, \end{aligned}$$

and

$$\begin{aligned} g^r(a_1,a_2,\ldots,a_n) &\geq \frac{p_n+p_{n-1}}{q_n+q_{n-1}}-\frac{\left(\frac{B^n}{2\alpha^n}+1\right)p_n+p_{n-1}}{\left(\frac{B^n}{2\alpha^n}+1\right)q_n+q_{n-1}} \\ &= \frac{\frac{B^n}{2\alpha^n}}{\left(\left(\frac{B^n}{2\alpha^n}+1\right)q_n+q_{n-1}\right)(q_n+q_{n-1})}. \end{aligned}$$

Thus,

$$G_n(a_1,a_2,\ldots,a_n) \geq \frac{1}{\left(\left(\frac{B^n}{\alpha^n}+1\right)q_n+q_{n-1}\right)(q_n+q_{n-1})},$$

and from (5.19), we have

$$G_n(a_1,\ldots,a_n) \geq \frac{1}{4}|J_n(a_1,\ldots,a_n)|.$$

Estimation of $\mu(B(x,r))$.

Now we are in a position to estimate the measure μ on a general ball $B(x,r)$. Fix $x \in \mathcal{F}_M(B)$ and let $B(x,r)$ be a ball centered at x with radius r small enough. There exists a unique sequence $a_1, \cdots, a_n$ such that $x \in J_n(a_1, \cdots, a_n)$ for each $n \geq 1$ and

$$G_{n+1}(a_1, \ldots, a_{n+1}) \leq r < G_n(a_1, \ldots, a_n).$$

It is clear, by the definition of G_n that $B(x,r)$ can intersect only one fundamental cylinder of order n, i.e., $J_n(a_1, \ldots, a_n)$.

Case I. $n = n_k$. Since in this case

$$|I_{n_k+1}(a_1, \ldots, a_{n_k+1})| = \frac{1}{q_{n_k+1}(q_{n_k+1} + q_{n_k})} \geq \frac{1}{6a_{n_{k+1}}^2 q_{n_k}^2} \geq \frac{\alpha^{2n_k}}{6B^{2n_k} q_{n_k}^2},$$

the number of fundamental cylinders of order $n_k + 1$ contained in $J_{n_k}(a_1, \ldots, a_{n_k})$ that the ball $B(x,r)$ intersects is at most

$$2r\frac{6B^{2n_k}}{\alpha^{2n_k}} q_{n_k}^2 + 2 \leq 24r\frac{B^{2n_k}}{\alpha^{2n_k}} q_{n_k}^2.$$

Therefore,

$$\begin{aligned}
\mu(B(x,r)) &\leq \min\Big\{\mu(J_{n_k}(a_1, \ldots, a_{n_k})), 24r\frac{B^{2n_k}}{\alpha^{2n_k}} q_{n_k}^2 \mu(J_{n_k+1}(a_1, \ldots, a_{n_k+1}))\Big\} \\
&\leq \mu(J_{n_k}(a_1, \ldots, a_{n_k})) \min\Big\{1, 48r\frac{B^{n_k}}{\alpha^{n_k}} q_{n_k}^2\Big\} \\
&\leq c|J_{n_k}(a_1, \ldots, a_{n_k})|^{s - \frac{6}{L} - \epsilon_0} \min\Big\{1, 48r\frac{B^{n_k}}{\alpha^{n_k}} q_{n_k}^2\Big\} \\
&\leq c\Big(\frac{2\alpha^{n_k}}{B^{n_k} q_{n_k}^2}\Big)^{s - \frac{6}{L} - \epsilon_0} (48r\frac{B^{n_k}}{\alpha^{n_k}} q_{n_k}^2)^{s - \frac{6}{L} - \epsilon_0} \\
&\leq c_0 r^{s - \frac{6}{L} - \epsilon_0}.
\end{aligned}$$

Here we use $\min\{a, b\} \leq a^{1-s} b^s$ for any $a, b > 0$ and $0 \leq s \leq 1$.

Case II. $n = n_k - 1$. In this case, since

$$|I_{n_k}(a_1, \ldots, a_{n_k})| = \frac{1}{q_{n_k}(q_{n_k} + q_{n_k-1})} \geq \frac{1}{6a_{n_k}^2 q_{n_k-1}^2} \geq \frac{1}{24\alpha^{2n_k} q_{n_k-1}^2},$$

the number of fundamental cylinders of order n_k contained in $J_{n_k-1}(a_1, \ldots, a_{n_k-1})$ that the ball $B(x,r)$ intersects is at most

$$48r\alpha^{2n_k} q_{n_k-1}^2 + 2 \leq 96r\alpha^{2n_k} q_{n_k-1}^2.$$

Therefore,

$$\mu(B(x,r)) \leq \min\Big\{\mu(J_{n_k-1}(a_1, \ldots, a_{n_k-1})), 96r\alpha^{2n_k} q_{n_k-1}^2 \mu(J_{n_k}(a_1, \ldots, a_{n_k}))\Big\}$$

$$\begin{aligned}
&\le \mu(J_{n_k-1}(a_1,\ldots,a_{n_k-1}))\min\left\{1, 48r\alpha^{n_k}q_{n_k-1}^2\right\}\\
&\le 12|J_{n_k-1}(a_1,\ldots,a_{n_k-1})|^{s-\frac{6}{L}-\epsilon_0}\min\left\{1, 48r\alpha^{n_k}q_{n_k-1}^2\right\}\\
&\le 12\Big(\frac{1}{2\alpha^{n_k}q_{n_k-1}^2}\Big)^{s-\frac{6}{L}-\epsilon_0}(48r\alpha^{n_k}q_{n_k-1}^2)^{s-\frac{6}{L}-\epsilon_0} \le c_0 r^{s-\frac{6}{L}-\epsilon_0}.
\end{aligned}$$

Case III. $n_{k-1}+1 \le n \le n_k-2$. As in this case $1 \le a_n(x) \le M$ and $|J_n(a_1,\ldots,a_n)| \asymp \frac{1}{q_n^2}$, we have

$$\begin{aligned}
\mu(B(x,r)) \le \mu(J_n(a_1,\ldots,a_n)) &\le c|J_n(a_1,\ldots,a_n)|^{s-\frac{6}{L}-\epsilon_0}\\
&\le c\left(\frac{1}{q_n^2}\right)^{s-\frac{6}{L}-\epsilon 0}\\
&\le c4M^2\left(\frac{1}{q_{n+1}^2}\right)^{s-\frac{6}{L}-\epsilon_0}\\
&\le c24M^2|J_{n+1}(a_1,\ldots,a_{n+1})|^{s-\frac{6}{L}-\epsilon 0}\\
&\le c48M^3(G_{n+1}(a_1,\ldots,a_{n+1}))^{s-\frac{6}{L}-\epsilon}\\
&\le c48M^3r^{s-\frac{6}{L}-\epsilon}.
\end{aligned}$$

Conclusion for the lower bound: Thus combining all the above cases and applying the mass distribution principle we have shown that $\dim_{\mathrm{H}}\mathcal{F}_M(B) \ge s-\frac{6}{L}-\epsilon_0$. Now letting $L\to\infty$, $M\to\infty$, by the choice of ϵ_0 for all large enough k and since $s<t_B$ is arbitrary, we have $s-\frac{6}{L}-\epsilon_0 \to t_B$.

Therefore,

$$\dim_{\mathrm{H}}\mathcal{F}(B) \ge \dim_{\mathrm{H}}\mathcal{F}_M(B) \ge t_B. \tag{5.23}$$

Taken together, results (5.7) and (5.23) complete the proof of the Theorem 5.1.1 for the case $1<B<\infty$.

Next we prove Theorem 5.1.1 for the case when $B=\infty$.

Case 2: $B=\infty$.

One can easily note that

$$a_n(x)a_{n+1}(x) \ge \Phi(n) \implies a_n(x) \ge \Phi(n)^{\frac{1}{2}} \text{ or } a_{n+1}(x) \ge \Phi(n)^{\frac{1}{2}}.$$

Thus

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset (\mathcal{V}_1(\Phi)\cup\mathcal{V}_2(\Phi)), \tag{5.24}$$

where

$$\mathcal{V}_1(\Phi) := \left\{x\in[0,1) : a_n(x) \ge \Phi(n)^{1/2} \text{ for infinitely many } n\in\mathbb{N}\right\}$$

and

$$\mathcal{V}_2(\Phi) := \left\{x\in[0,1) : a_{n+1}(x) \ge \Phi(n)^{1/2} \text{ for infinitely many } n\in\mathbb{N}\right\}.$$

2a. If $b = 1$, then for any $\delta > 0$, we have $\frac{\log\log\Phi(n)}{n} \leq \log(1+\delta)$ that is $\Phi(n) \leq e^{(1+\delta)^n}$ for infinitely many $n \in \mathbb{N}$. Since

$$\left\{x \in [0,1) : a_n(x) \geq e^{(1+\delta)^n} \text{ for all sufficiently large } n \in \mathbb{N}\right\} \subset \mathcal{F}(\Phi).$$

Therefore, by using Lemma 2.1.3,

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) \geq \lim_{\delta \to 0} \frac{1}{1+1+\delta} = \frac{1}{2}.$$

Note that as $B = \infty$, for any $C > 1$, we have $\Phi(n) \geq C^n$ for all sufficiently large $n \in \mathbb{N}$. Thus by (5.24),

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \{x \in [0,1) : a_n(x) \geq C^n \text{ for infinitely many } n \in \mathbb{N}\}.$$

By Proposition 5.0.3 and Theorem 5.0.2, we have

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) \leq \lim_{C \to \infty} s_C = \frac{1}{2}.$$

2b. If $1 < b < \infty$ then for any $\delta > 0$, we have $\frac{\log\log\Phi(n)}{n} \leq \log(b+\delta)$, that is $\Phi(n) \leq e^{(b+\delta)^n}$ for infinitely many $n \in \mathbb{N}$, whereas $\Phi(n) \geq e^{(b-\delta)^n}$ for all sufficiently large $n \in \mathbb{N}$. Since

$$\left\{x \in [0,1) : a_n(x) \geq e^{(1+\delta)^n} \text{ for all sufficiently large } n \in \mathbb{N}\right\} \subset \mathcal{F}(\Phi).$$

Therefore, by using Lemma 2.1.3

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) \geq \lim_{\delta \to 0} \frac{1}{1+b+\delta} = \frac{1}{1+b}.$$

Further note that from the definition of the set $\mathcal{V}_i(\Phi)$ where $i = 1, 2$, it is clear that

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \left\{x \in [0,1) : a_n(x) \geq e^{\frac{1}{2}(b-\delta)^{(n-1)}} \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

By Lemma 2.1.3,

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) \leq \lim_{\delta \to 0} \frac{1}{1+b-\delta} = \frac{1}{1+b}.$$

2c. If $b = \infty$ then by using the same argument as for showing the upper bound in case 2b, for any $C > 1$, we have $\Phi(n) \geq e^{C^n}$ for all sufficiently large $n \in \mathbb{N}$. By (5.24), we have

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \left\{x \in [0,1) : a_n(x) \geq e^{C^n} \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

Also, by Proposition 5.0.3 and Theorem 5.0.2,

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) \leq \lim_{C \to \infty} \frac{1}{C+1} = 0.$$

This completes the proof of Theorem 5.1.1. ∎

Finally, we remark that it is possible to generalise the set $\mathcal{F}(\Phi)$ for any $d \geq 2$, to the following

$$\mathcal{F}_d(\Phi) = \left\{ x \in [0,1) : \begin{array}{c} \prod_{k=1}^{d-1} a_{n+k-1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ \prod_{k=1}^{d-1} a_{n+k-1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

By following the same method as we have used for the proof of Theorem 5.1.1, we can show that.

Theorem 5.2.1 *Let $\Phi : \mathbb{N} \to (1, \infty)$ be any function with $\lim_{n\to\infty} \Phi(n) = \infty$. Define B, b as in Theorem 5.1.1. Then*

1. $\dim_{\mathrm{H}} \mathcal{F}_d(\Phi) = \inf\{s \geq 0 : \mathsf{P}(T, -g_d \log B - s \log |T'|) \leq 0\}$ *when* $1 < B < \infty$, *where* $g_1 = s$, $g_d = \frac{s g_{d-1}(s)}{1-s+g_{d-1}(s)}$ *for* $d \geq 2$;
2. $\dim_{\mathrm{H}} \mathcal{F}_d(\Phi) = \frac{1}{(1+b)}$ *when* $B = \infty$.

More general setup

In the next chapter we will give a natural generalisation of classical Jarník–Besicovitch Theorem (1928, 1934).

Chapter 6

Generalised metrical properties of continued fractions

In this chapter we introduce the set of points $x \in [0, 1)$ for which the product of an arbitrary block of consecutive partial quotients in their continued fraction expansion are growing. In fact we aim to compute the Hausdorff dimension of the set of $x \in [0, 1)$ such that for any $r \in \mathbb{N}$,

$$\log(a_{n+1}(x) \cdots a_{n+r}(x)) \geq \tau(x)(h(x) + \cdots + h(T^{n-1}(x)))$$

for infinitely many $n \in \mathbb{N}$, where h and τ are positive continuous functions, T is the Gauss map and $a_n(x)$ denote the nth partial quotient of x in its continued fraction expansion. Later in this chapter we will see that by appropriate choices of r, $\tau(x)$ snd $h(x)$ we obtain the classical Jarník–Besicovitch Theorem as well as more recent results by Wang–Wu–Xu [46], Wang–Wu [45], Huang–Wu–Xu [25] and Hussain–Kleinbock–Wadleigh–Wang [27].

Recall from Chapter 1, that the metrical aspect of the theory of continued fractions has been very well studied due to its close connections with Diophantine approximation. From Section 1.1 it can be observed that this theory can be viewed as arising from the Gauss transformation (1.1).

For the requirement of this chapter we reformulate the set of τ-well approximable points as follows, calling it the Jarník–Besicovitch set,

$$\left\{x \in [0, 1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

Clearly, Jarník–Besicovitch Theorem [29, 8] gives the Hausdorff dimension of this set to be $\frac{2}{\tau}$, for any $\tau \geq 2$.

Recall that for any irrational $x \in [0, 1)$, the irrationality exponent of x is defined as

$$\vartheta(x) := \sup\left\{\tau : \left|x - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

From Theorem 1.1.1, it is known that $\vartheta(x) \geq 2$ for any irrational $x \in [0,1)$. Moreover for any $\tau \geq 2$, Jarník–Besicovitch Theorem [29, 8] states that

$$\dim_{\mathrm{H}}\{x \in [0,1) : \vartheta(x) \geq \tau\} = \dim_{\mathrm{H}}\{x \in [0,1) : \vartheta(x) = \tau\} = \frac{2}{\tau}.$$

Observe that the exponent τ in the above sets is constant. Barral–Seuret [4] generalised Jarník–Besicovitch Theorem by considering the set of points for which the irrationality exponent is not fixed in advance but may vary with x in a continuous way. More precisely, Barral–Seuret [4] showed that for a continuous function $\tau(x)$,

$$\begin{aligned}\dim_{\mathrm{H}}\{x \in [0,1) : \vartheta(x) \geq \tau(x)\} &= \dim_{\mathrm{H}}\{x \in [0,1) : \vartheta(x) = \tau(x)\} \\ &= \frac{2}{\min\{\tau(x) \ : \ x \in [0,1]\}}.\end{aligned}$$

They called such a set the localised Jarník–Besicovitch set. Their result was further generalised by Wang–Wu–Xu [46] who refashioned the problem in terms of continued fractions and took a dynamical approach. To refer their result, first denote by $S_n f(x) := f(x) + \cdots + f(T^{n-1}(x))$ the ergodic sum of any function f and then by using the facts (1.4), (2.6), and (2.7), the Jarník–Besicovitch set in terms of growth rate of partial quotients can be restated as,

$$J(\tau) := \left\{x \in [0,1) : a_n(x) \geq e^{\left(\frac{\tau-2}{2}\right)\cdot S_n(\log|T'(x)|)} \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

Note that the set $J(\tau)$, in terms of entries of continued fractions, contains the approximating function that involves the ergodic sum

$$S_n(\log|T'(x)|) = \log|T'(x)| + \cdots + \log|T'(T^{n-1}(x))|$$

and this sum is growing fast as $n \to \infty$. Therefore having the approximating function in terms of the ergodic sum and the fact that partial quotients of any real number $x \in [0,1)$ completely determines its Diophantine properties, the Jarník–Besicovitch set (and its related variations which we will see in this chapter) in terms of the growth rate of partial quotients gives us better approximation results. In fact, Wang–Wu–Xu [46] introduced the generalised version of $J(\tau)$ as

$$J(\tau; h) := \left\{x \in [0,1) : a_n(x) \geq e^{\tau(x)\cdot S_n h(x)} \text{ for infinitely many } n \in \mathbb{N}\right\},$$

where $h(x)$ and $\tau(x)$ are positive continuous functions defined on $[0,1]$. They called such points the localised $(\tau; h)$ approximable points. Further, they proved that

$$\dim_{\mathrm{H}} J(\tau, h) = s_{\mathbb{N}}^{(1)} := \inf\{s \geq 0 : \mathsf{P}(T, -s\tau_{\min}h - s\log|T'|) \leq 0\},$$

where $\tau_{\min} = \min\{\tau(x) : x \in [0,1]\}$, P denotes the pressure function and T' is the derivative of the Gauss map T.

In this chapter we introduce the set of points $x \in [0,1)$ for which the product of an arbitrary block of consecutive partial quotients, in their continued fraction expansion, are growing. In fact we determine the size of such a set in terms of Hausdorff dimension. Motivation for considering the growth of product of consecutive partial quotients arose from the work of Kleinbock–Wadleigh [34] where they considered improvements to Dirichlet's Theorem. We refer the reader to [3, 2, 27, 33, 34, 35] for comprehensive metric theory associated with the set of points improving Dirichlet's Theorem.

6.1 Statement of main result

We prove the following main result of this chapter. Note that tempered distortion property (2.9) is defined in Chapter 2.

Theorem 6.1.1 *Let $h : [0,1] \to (0,\infty)$ and $\tau : [0,1] \to [0,\infty)$ be positive continuous functions with h satisfying the tempered distortion property. For $r \in \mathbb{N}$ define the set*

$$\mathcal{R}_r(\tau; h) := \left\{ x \in [0,1) : \prod_{d=1}^{r} a_{n+d}(x) \geq e^{\tau(x)\cdot S_n h(x)} \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Then

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h) = s_{\mathbb{N}}^{(r)} := \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min} h - s\log|T'|) \leq 0\},$$

where $\tau_{\min} = \min\{\tau(x) : x \in [0,1]\}, g_1(s) = s$ and $g_r(s) = \frac{s g_{r-1}(s)}{1-s+g_{r-1}(s)}$ for all $r \geq 2$. ■

Theorem 6.1.1 is more general as for different $\tau(x)$ and $h(x)$ it implies various classical results as we now see.

- When $r = 1$, $\tau(x) = c$ where c is a constant and $h(x) = \log|T'|$, then we obtain the classical Jarník–Besicovitch Theorem [8, 29] .

  Corollary 6.1.2 *For any $\tau \geq 2$,*

  $$\dim_{\mathrm{H}} J(\tau) = \frac{2}{\tau}.$$ ■

- When $r = 1$, we obtain the result by Wang–Wu–Xu [46] .

  Corollary 6.1.3

  $$\dim_{\mathrm{H}} \left\{ x \in [0,1) : a_{n+1}(x) \geq e^{\tau(x) S_n h(x)} \text{ for infinitely many } n \in \mathbb{N} \right\} = s_{\mathbb{N}}^{(1)}.$$ ■

- When $r = 1, \tau(x) = 1$ and $h(x) = \log B$, we obtain Theorem 5.0.2 i.e. the result by Wang–Wu [45].

Corollary 6.1.4 *For any $B > 1$,*

$$\dim_{\mathrm{H}} \{x \in [0,1) : a_{n+1}(x) \geq B^n \text{ for infinitely many } n \in \mathbb{N}\} = s_B^{(1)}$$

where

$$s_B^{(1)} = \inf\{s \geq 0 : \mathsf{P}(T, -s\log B - s\log|T'|) \leq 0\}.$$

- When $\tau(x) = 1$ and $h(x) = \log B$, we obtain the result by Huang–Wu–Xu [25].

Corollary 6.1.5 *For any $B > 1$,*

$$\dim_{\mathrm{H}} \{x \in [0,1) : a_{n+1}(x) \cdots a_{n+r}(x) \geq B^n \text{ for infinitely many } n \in \mathbb{N}\} = s_B^{(r)},$$

where

$$s_B^{(r)} = \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\log B - s\log|T'|) \leq 0\}.$$

- When $r = 2$, $\tau(x) = c$ where c is a constant and $h(x) = \log|T^{'}|$, we obtain the result by Hussain–Kleinbock–Wadleigh–Wang [27].

Corollary 6.1.6

$$\dim_{\mathrm{H}} \left\{x \in [0,1) : a_{n+1}(x)a_{n+2}(x) \geq q_{n+1}^{\tau+2}(x) \text{ for infinitely many } n \in \mathbb{N}\right\} = \frac{2}{2+\tau}.$$

The pressure function and $s_{\mathbb{N}}^{(r)}$

Consider a finite or infinite subset $\mathcal{A}$ of the set of natural numbers and for every $n \geq 1$ and $s \geq 0$, let

$$f_{n,\mathcal{A}}(s) = \sum_{a_1,\ldots,a_n \in \mathcal{A}} \frac{1}{e^{g_r(s)\tau_{\min} S_n h(z)} q_n^{2s}}, \tag{6.1}$$

where $z \in I_n(a_1, \cdots, a_n)$ and $g_r(s)$ is defined by the formula

$$g_1(s) = s \text{ and } g_r(s) = \frac{s g_{r-1}(s)}{1 - s + g_{r-1}(s)} \text{ for all } r \geq 2. \tag{6.2}$$

It can be easily checked that for any $s \in (\frac{1}{2}, 1)$ we have $g_{r+1}(s) \leq g_r(s)$, for all $r \geq 1$.

For the requirement of this chapter we consider a particular potential, that is,

$$\varphi_s(x) := -g_r(s)\tau_{\min} h - s\log|T'(x)|.$$

From the definition of pressure function (2.10) and using equations (2.7), (6.1) and (6.2) we have

$$\mathsf{P}_{\mathcal{A}}(T, \varphi_s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\ldots,a_n \in \mathcal{A}} \frac{1}{e^{g_r(s)\tau_{\min} S_n h(z)} q_n^{2s}}.$$

Define

$$s_{n,\mathcal{A}}^{(r)} = \inf\left\{s \geq 0 : f_{n,\mathcal{A}}(s) \leq 1\right\},$$

and let

$$\begin{aligned} s_{\mathcal{A}}^{(r)} &= \inf\{s \geq 0 : \mathsf{P}_{\mathcal{A}}(T, -g_r(s)\tau_{\min}h - s\log|T'|) \leq 0\}, \\ s_{\mathbb{N}}^{(r)} &= \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min}h - s\log|T'|) \leq 0\}. \end{aligned}$$

When $\mathcal{A}$ is a finite subset of $\mathbb{N}$, then $s_{n,\mathcal{A}}^{(r)}$ and $s_{\mathcal{A}}^{(r)}$ are the unique solutions to $f_{n,\mathcal{A}}(s) = 1$ and $\mathsf{P}_{\mathcal{A}}(T, -g_r(s)\tau_{\min}h - s\log|T'(x)|) = 0$, respectively (for details see [45]). If $\mathcal{A} = \{1, \cdots, M\}$ for any $M \in \mathbb{N}$, write $s_{n,M}^{(r)}$ for $s_{n,\mathcal{A}}^{(r)}$ and $s_M^{(r)}$ for $s_{\mathcal{A}}^{(r)}$.

From Proposition 2.2.2 and since the potential φ_s satisfies the variation property we have the following result.

Corollary 6.1.7 *For any integer $r \geq 1$,*

$$s_{\mathbb{N}}^{(r)} = \sup\{s_{\mathcal{A}}^{(r)} : \mathcal{A} \textit{ is a finite subset of } \mathbb{N}\}.$$

Furthermore, the dimensional term $s_{\mathbb{N}}^{(r)}$ is continuous with respect to φ_s, i.e.,

$$\lim_{\epsilon \to 0} \inf\{s \geq 0 : \mathsf{P}_{\mathcal{A}}(T, \varphi_s + \epsilon) \leq 0\} = \inf\{s \geq 0 : \mathsf{P}_{\mathcal{A}}(T, \varphi_s) \leq 0\}. \tag{6.3}$$

■

From the definition of pressure function and by Corollary 6.1.7 for any $M \in \mathbb{N}$ and $0 < \epsilon < 1$, we have

$$\lim_{n\to\infty} s_{n,M}^{(r)} = s_M^{(r)}, \quad \lim_{n\to\infty} s_{n,\mathbb{N}}^{(r)} = s_{\mathbb{N}}^{(r)}, \quad \lim_{M\to\infty} s_M^{(r)} = s_{\mathbb{N}}^{(r)} \text{ and } |s_{n,M}^{(r)} - s_M^{(r)}| \leq \epsilon. \tag{6.4}$$

6.2 Proof of Theorem 6.1.1

The proof of Theorem 6.1.1 is divided into two main parts:
(i) the upper bound for $\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h)$ and
(ii) the lower bound for $\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h)$.

Proof of Theorem 6.1.1: the upper bound

For the upper bound we will find a natural covering for the set $\mathcal{R}_r(\tau; h)$. To do this, recall that h is assumed to be a positive continuous function satisfying tempered distortion property (2.9). Consequently,

$$\frac{1}{n}\sum_{j=1}^{n} \mathrm{Var}_j(h) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus for any fixed $\lambda > 0$ there exist $N(\lambda) \in \mathbb{N}$ such that for any $n \geq N(\lambda)$ we have $\sum_{j=1}^{n} \operatorname{Var}_j(h) \leq n\lambda$. Therefore, for any $x, z \in [0,1)$ with $I_n(x) = I_n(z)$ we have

$$\begin{aligned}
|S_n h(x) - S_n h(z)| &= \left| \sum_{j=0}^{n-1} h(T^j x) - \sum_{j=0}^{n-1} h(T^j z) \right| \\
&\leq \sum_{j=0}^{n-1} \left| h(T^j x) - h(T^j z) \right| \\
&\leq \sum_{j=0}^{n-1} \operatorname{Var}_{n-j}(h) \leq n\lambda.
\end{aligned}$$

Then it follows that

$$\mathcal{R}_r(\tau; h) \subset \mathcal{C}_r(\tau)$$

where

$$\mathcal{C}_r(\tau) := \left\{ x \in [0,1) : \prod_{d=1}^{r} a_{n+d}(x) \geq e^{\tau_{\min} S_n(h-\lambda)(z)} \text{ for infinitely many } n \in \mathbb{N} \right\}$$

and $z \in I_n(a_1, \cdots, a_n)$. Thus for the upper bound of $\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h)$ it is sufficient to calculate the upper bound for $\dim_{\mathrm{H}} \mathcal{C}_r(\tau)$ i.e., to show

$$\dim_{\mathrm{H}} \mathcal{C}_r(\tau) \leq \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min}(h-\lambda) - s \log |T'|) \leq 0\}, \tag{6.5}$$

which we will prove by induction on r.

For $r = 1$, the result is proved by Wang–Wu–Xu [46].

Suppose that (6.5) is true for $r = k$. We need to show that (6.5) holds for $r = k+1$. Note that

$$\begin{aligned}
\mathcal{C}_{k+1}(\tau) \subseteq & \left\{ x \in [0,1) : \prod_{d=1}^{k} a_{n+d}(x) \geq e^{\tau_{\min} S_n(h-\lambda)(z)} \text{ for infinitely many } n \in \mathbb{N} \right\} \\
& \cup \left\{ x \in [0,1) : \begin{array}{l} 1 \leq \prod_{d=1}^{k} a_{n+d}(x) \leq e^{\tau_{\min} S_n(h-\lambda)(z)}, \\ a_{n+k+1}(x) \geq \dfrac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}(x)} \text{ for infinitely many } n \in \mathbb{N} \end{array} \right\}.
\end{aligned}$$

Further, for any $1 < \gamma \leq e$,

$$\mathcal{C}_{k+1}(\tau) \subseteq \mathcal{I}(\tau) \cup \mathcal{J}(\tau),$$

where

$$\mathcal{I}(\tau) := \left\{ x \in [0,1) : \prod_{d=1}^{k} a_{n+d}(x) \geq \gamma^{\tau_{\min} S_n(h-\lambda)(z)} \quad \text{for infinitely many } n \in \mathbb{N} \right\}$$

and

$$\mathcal{J}(\tau) := \left\{ x \in [0,1) : \begin{array}{c} 1 \leq \prod_{d=1}^{k} a_{n+d}(x) \leq \gamma^{\tau_{\min} S_n(h-\lambda)(z)}, \\ a_{n+k+1}(x) \geq \dfrac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}(x)} \text{ for infinitely many } n \in \mathbb{N} \end{array} \right\}.$$

Therefore,

$$\dim_{\mathrm{H}} \mathcal{C}_{k+1}(\tau) \leq \inf_{1<\gamma\leq e} \max \{\dim_{\mathrm{H}} \mathcal{I}(\tau), \dim_{\mathrm{H}} \mathcal{J}(\tau)\}.$$

By using induction hypothesis and since $\gamma^{\tau_{\min} S_n(h-\lambda)(z)} \leq e^{\tau_{\min} S_n(h-\lambda)(z)}$, we have

$$\dim_{\mathrm{H}} \mathcal{I}(\tau) \leq t_\gamma^k := \inf\{s \geq 0 : \mathsf{P}(T, -g_k(s)\tau_{\min}(h-\lambda)\log\gamma - s\log|T'|) \leq 0\}.$$

For the upper bound of $\dim_{\mathrm{H}} \mathcal{J}(\tau)$ we proceed by finding a natural covering for this set. In terms of lim sup nature of the set $\mathcal{J}(\tau)$, can be rewritten as

$$\mathcal{J}(\tau) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{J}^*(\tau) := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ x \in [0,1) : \begin{array}{c} 1 \leq \prod_{d=1}^{k} a_{n+d}(x) \leq \gamma^{\tau_{\min} S_n(h-\lambda)(z)}, \\ a_{n+k+1}(x) \geq \dfrac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}(x)} \end{array} \right\}.$$

Thus for each $n \geq N$, the cover for $\mathcal{J}^*(\tau)$ will serve as a natural cover for $\mathcal{J}(\tau)$. Clearly,

$$\mathcal{J}^*(\tau) \subseteq \bigcup_{a_1,\cdots,a_n \in \mathbb{N}} \bigcup_{1 \leq \prod_{d=1}^{k} a_{n+d} \leq \gamma^{\tau_{\min} S_n(h-\lambda)(z)}} J_{n+k}(a_1, \cdots, a_{n+k})$$

where

$$J_{n+k}(a_1, \cdots, a_{n+k}) = \bigcup_{a_{n+k+1} \geq \frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}}} I_{n+k+1}(a_1, \cdots, a_{n+k+1}).$$

By using (2.2), we have

$$\begin{aligned} |J_{n+k}(a_1, \ldots, a_{n+k})| &= \sum_{a_{n+k+1} \geq \frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}}} |I_{n+k+1}(a_1, \ldots, a_{n+k}, a_{n+k+1})| \\ &\leq \sum_{a_{n+k+1} \geq \frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}}} \frac{1}{a_{n+k+1} q_{n+k}^2(a_1, \cdots, a_{n+k})} \\ &\asymp \sum_{a_{n+k+1} \geq \frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{\prod_{d=1}^{k} a_{n+d}}} \frac{1}{a_{n+k+1} \left(\prod_{d=1}^{k} a_{n+d}\right)^2 q_n^2(a_1, \cdots, a_n)} \\ &\asymp \frac{1}{e^{\tau_{\min} S_n(h-\lambda)(z)} \left(\prod_{d=1}^{k} a_{n+d}\right) q_n^2(a_1, \cdots, a_n)}. \end{aligned} \tag{6.6}$$

Fixing $\delta > 0$ and taking the $(s+\delta)$-volume of the cover of $J^*(\tau)$, we obtain

$$\begin{aligned}
&\sum_{a_1,\cdots,a_n\in\mathbb{N}}\ \sum_{1\le a_{n+1}\cdots a_{n+k}\le\gamma^{\tau_{\min}S_n(h-\lambda)(z)}}\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}(a_{n+1}\cdots a_{n+k})q_n^2(z)}\right)^{s+\delta}\\
\le&\sum_{a_1,\cdots,a_n\in\mathbb{N}}\ \sum_{1\le a_{n+1}\cdots a_{n+k}\le\gamma^{\tau_{\min}S_n(h-\lambda)(z)}}\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}(a_{n+1}\cdots a_{n+k})q_n^2(z)}\right)^{s}e^{-\delta\tau_{\min}S_n(h-\lambda)(z)}\\
\asymp&\sum_{a_1,\cdots,a_n\in\mathbb{N}}\frac{(\log\gamma^{\tau_{\min}S_n(h-\lambda)(z)})^{k-1}}{(k-1)!}\gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)}\cdot\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\right)^{s}e^{-\delta\tau_{\min}S_n(h-\lambda)(z)}\\
\le&\sum_{a_1,\cdots,a_n\in\mathbb{N}}\frac{(\log e^{\tau_{\min}S_n(h-\lambda)(z)})^{k-1}}{(k-1)!}\gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)}\cdot\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\right)^{s}e^{-\delta\tau_{\min}S_n(h-\lambda)(z)}\\
&\qquad\le\sum_{a_1,\cdots,a_n\in\mathbb{N}}\gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)}\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\right)^{s}.
\end{aligned}$$

Therefore, the $(s+\delta)$-dimensional Hausdorff measure of $\mathcal{J}(\tau)$ is

$$\begin{aligned}
&\mathcal{H}^{s+\delta}(\mathcal{J}(\tau))\\
\le&\liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{a_1,\cdots,a_n\in\mathbb{N}}\ \sum_{1\le\prod_{d=1}^{k}a_{n+d}\le\gamma^{\tau_{\min}S_n(h-\lambda)(z)}}\\
&\qquad\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}\left(\prod_{d=1}^{k}a_{n+d}\right)q_n^2(z)}\right)^{s+\delta}\\
\le&\liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{a_1,\cdots,a_n\in\mathbb{N}}\gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)}\left(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\right)^{s}.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, it follows that

$$\dim_{\mathrm{H}}\mathcal{J}(\tau)\le u_\gamma^{k+1},$$

where u_γ^{k+1} is defined as

$$\inf\{s\ \ge\ 0\ :\ \mathsf{P}(T,(1-s)\tau_{\min}(h-\lambda)\log\gamma-s\tau_{\min}(h-\lambda)-s\log|T'|)\ \le\ 0\}.$$

Hence,

$$\dim_{\mathrm{H}}\mathcal{C}_{k+1}(\tau)\le\inf_{1<\gamma\le e}\max\{t_\gamma^k,u_\gamma^{k+1}\}.$$

As the pressure function $\mathsf{P}(T,\varphi_s)$ is increasing with respect to the potential φ_s we know t_γ^k is increasing and u_γ^{k+1} is decreasing with respect to γ. Therefore the infimum is obtained at the value γ where

$$-g_k(s)\tau_{\min}(h-\lambda)\log\gamma-s\log|T'|=(1-s)\tau_{\min}(h-\lambda)\log\gamma-s\tau_{\min}(h-\lambda)-s\log|T'|.$$

This implies that

$$\gamma^{(1-s)\tau_{\min}(h-\lambda)}e^{-s\tau_{\min}(h-\lambda)}=\gamma^{-g_k(s)\tau_{\min}(h-\lambda)}$$

$$\begin{aligned}
&\iff -\frac{s}{(1-s)+g_k(s)} = -\log\gamma \\
&\iff -g_{k+1}(s) = -g_k(s)\log\gamma.
\end{aligned}$$

Hence,

$$\dim_{\mathrm{H}} \mathcal{C}_{k+1}(\tau) \leq \inf\{s \geq 0 : \mathsf{P}(T, -g_{k+1}(s)\tau_{\min}(h-\lambda) - s\log|T'|) \leq 0\}.$$

Consequently, for any $r \geq 1$, we obtain

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h) \leq \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min}(h-\lambda) - s\log|T'|) \leq 0\}.$$

By (6.3) and letting $\lambda \to 0$ we obtained the desired result.

Proof of Theorem 6.1.1: the lower bound

For the lower bound, again our strategy is to first construct an appropriate Cantor subset $\mathcal{E}_\infty$ of $\mathcal{R}_r(\tau; h)$, then distribute the measure $\mu > 0$ on $\mathcal{E}_\infty$ and obtain the Hölder exponent. Lastly, we apply the mass distribution principle, i.e., Proposition 2.1.4.

Cantor subset:

Fix $\frac{1}{2} < s < s_{\mathbb{N}}^{(r)}$ and choose $1 \leq \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{r-2} \leq e$ in a way such that

$$\log\gamma_i = \frac{g_r(s)(1-s)^i}{s^{i+1}} \text{ for all } 0 \leq i \leq r-2. \tag{6.7}$$

Moreover we have the following lemma which we will prove by induction on r.

Lemma 6.2.1 For any $r \geq 1$,

$$g_r(s) = \frac{s^r(2s-1)}{s^r-(1-s)^r} \tag{6.8}$$

satisfies the recursive relation defined in (6.2).

Proof: When $r = 1$, it is clear from (6.2) that

$$g_1(s) = s = \frac{s(2s-1)}{s-(1-s)}.$$

Suppose (6.8) is true for $r = k$, then for $r = k+1$

$$\begin{aligned}
g_{k+1}(s) &= \frac{sg_k(s)}{1-s+g_k(s)} \text{ by (6.2)} \\
&= \frac{s\frac{s^k(2s-1)}{s^k-(1-s)^k}}{1-s+\frac{s^k(2s-1)}{s^k-(1-s)^k}} \text{ (by induction hypothesis)} \\
&= \frac{s^{k+1}(2s-1)}{s^k-(1-s)^k-s^{k+1}+s(1-s)^k+(2s-1)s^k} \\
&= \frac{s^{k+1}(2s-1)}{s^{k+1}-(1-s)^k(1-s)} = \frac{s^{k+1}(2s-1)}{s^{k+1}-(s-1)^{k+1}}.
\end{aligned}$$

■

Therefore (6.8) is true for $r = k+1$.

Thus by using (6.7) and (6.8) it is easy to check that the following equality holds.

$$\begin{aligned}\log\gamma_0^{-s} = \log(\gamma_0^{1-s}(\gamma_0\gamma_1)^{-s}) = \cdots &= \log((\gamma_0\cdots\gamma_{r-3})^{1-s}(\gamma_0\cdots\gamma_{r-2})^{-s}) \\ &= \log(\gamma_0\cdots\gamma_{r-2})^{1-s} - s = -g_r(s).\end{aligned} \tag{6.9}$$

Further, let $\epsilon > 0$ and $M \in \mathbb{N}$. Fix an irrational z_0 and an integer t_0 such that for any $z \in I_n(z_0)$ with $n \geq t_0$, we have

$$\tau(z) \leq \min\{\tau_{\min}(1+\epsilon), \tau_{\min}+\epsilon\}.$$

Next define two integer sequences $\{t_j\}_{j\geq 1}$ and $\{m_j\}_{j\geq 1}$ recursively where $\{m_j\}_{j\geq 1}$ is defined to be a largely sparse integer sequence tending to infinity. For each $j \geq 1$, define $t_j = t_0 + j$ and set $n_j = (n_{j-1} + (r-1)) + t_j + m_j + 1$.

Now we construct the Cantor subset '$\mathcal{E}_\infty$' level by level. We start by defining the zero level.

Level 0. Let $n_0 + (r-1) \geq t_2$. Define

$$\nu^{(0,r-1)} = (a_1(z_0), a_2(z_0), \cdots, a_{n_0+(r-1)}(z_0)).$$

Then the zero level '$\mathcal{E}_{0,r-1}$' of the Cantor set $\mathcal{E}_\infty$ is defined as

$$\mathcal{E}_{0,r-1} := \mathcal{F}_0 = \{I_{n_0+(r-1)}(\nu^{(0,r-1)})\}.$$

Level 1. Note that $n_1 = (n_0 + (r-1)) + t_1 + m_1 + 1$. Let us define the collection of basic cylinders of order $n_1 - 1$:

$$\mathcal{F}_1 = \{I_{n_1-1}(\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, b_1^{(1)}, \cdots, b_{m_1}^{(1)}) : 1 \leq b_1^{(1)}, \cdots, b_{m_1}^{(1)} \leq M\}.$$

For each $I_{n_1-1}(w^{(1)}) \in \mathcal{F}_1$ where $w^{(1)} := (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, b_1^{(1)}, \cdots, b_{m_1}^{(1)})$, define the collection of sub-cylinders of order n_1:

$$\begin{aligned}\mathcal{E}_{1,0}(w^{(1)}) := \{&I_{n_1}(\nu^{(1,0)}) := I_{n_1}(w^{(1)}, a_{n_1}) : \\ &\gamma_0^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \leq a_{n_1} < 2\gamma_0^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}\end{aligned} \tag{6.10}$$

where $z_1 \in I_{n_1-(n_0+(r-1))-1}(\nu^{(0,r-1)}|_{t_1}, b_1^{(1)}, \cdots, b_{m_1}^{(1)})$.

Let $I_{n_1} = I_{n_1}(\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, b_1^{(1)}, \cdots, b_{m_1}^{(1)}, a_{n_1}) \in \mathcal{E}_{1,0}(w^{(1)})$. The choice of z_1 indicates that for any $x \in I_{n_1}$ the continued fraction representations of z_1 and x share prefixes up to t_1th partial quotients. Hence $\tau(z_1)$ is close to $\tau(x)$ by the continuity of τ. Further, it can be easily checked that $S_{n_1-(n_0+(r-1))-1}h(z_1) \sim S_{n_1-1}h(x)$, (here '$\sim$' denotes the asymptotic equality of two functions). Consequently,

$$\begin{aligned}&\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1) \sim \tau(x)S_{n_1-1}h(x) \\ \Longrightarrow \quad &\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)\log\gamma_0 \sim \tau(x)S_{n_1-1}h(x)\log\gamma_0\end{aligned}$$

$$\implies \qquad \log \gamma_0^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \sim \log \gamma_0^{\tau(x)S_{n_1-1}h(x)}$$
$$\implies \qquad \log a_{n_1} \sim \log \gamma_0^{\tau(x)S_{n_1-1}h(x)} \text{ from } (6.10)\ .$$

Thus, $a_{n_1}(x) \sim \gamma_0^{\tau(x)S_{n_1-1}h(x)}$.

Next for each $I_{n_1}(\nu^{(1,0)}) \in \mathcal{E}_{1,0}(w^{(1)})$, define

$$\mathcal{E}_{1,1}(\nu^{(1,0)}) := \{I_{n_1+1}(\nu^{(1,1)}) := I_{n_1+1}(\nu^{(1,0)}, a_{n_1+1}) : \gamma_1^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \leq a_{n_1+1} < 2\gamma_1^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}.$$

Continuing in this way for each $I_{n_1+(r-3)}(\nu^{(1,r-3)}) \in \mathcal{E}_{1,r-3}(\nu^{(1,r-4)})$ collect a family of sub-cylinders of order $n_{1+(r-2)}$:

$$\mathcal{E}_{1,r-2}(\nu^{(1,r-3)}) := \{I_{n_1+(r-2)}(\nu^{(1,r-2)}) = I_{n_1+(r-2)}(\nu^{(1,r-3)}, a_{n_1+(r-2)}) : \gamma_{r-2}^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \leq a_{n_1+(r-2)} < 2\gamma_{r-2}^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}.$$

Further for each $I_{n_1+(r-2)}(\nu^{(1,r-2)}) \in \mathcal{E}_{1,r-2}(\nu^{(1,r-3)})$ collect a family of sub-cylinders of order $n_{1+(r-1)}$:

$$\mathcal{E}_{1,r-1}(\nu^{(1,r-2)}) := \{I_{n_1+(r-1)}(\nu^{(1,r-1)}) = I_{n_1+(r-1)}(\nu^{(1,r-2)}, a_{n_1+(r-1)}) : \left(\frac{e}{\gamma_0 \cdots \gamma_{r-2}}\right)^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \leq a_{n_1+(r-1)} < 2\left(\frac{e}{\gamma_0 \cdots \gamma_{r-2}}\right)^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}.$$

Then the first level '$\mathcal{E}_{1,r-1}$' of the Cantor set $\mathcal{E}_\infty$ is defined as

$$\mathcal{E}_{1,r-1} = \{I_{n_1+(r-1)}(\nu^{(1,r-1)}) \in \mathcal{E}_{1,r-1}(\nu^{(1,r-2)}) : I_{n_1+i}(\nu^{(1,i)}) \in \mathcal{E}_{1,i}(\nu^{(1,i-1)}) \text{ for } 1 \leq i \leq r-2; I_{n_1}(\nu^{(1,0)}) \in \mathcal{E}_{1,0}(w^{(1)}); I_{n_1-1}(w^{(1)}) \in \mathcal{F}_1\}.$$

Level j.

Suppose that $\mathcal{E}_{j-1,r-1}$ that is the $(j-1)$th level has been constructed. Clearly, $\mathcal{E}_{j-1,r-1}$ consists of the collection of basic cylinders which are of order $n_{j-1}+(r-1)$. Recall that $n_j = (n_{j-1}+(r-1))+t_j+m_j+1$. For each $I_{n_{j-1}+(r-1)}(\nu^{(j-1,r-1)}) \in \mathcal{E}_{j-1,r-1}$ define the collections of sub-cylinders of order n_j-1:

$$\mathcal{F}_j(I_{n_{j-1}+(r-1)}(\nu^{(j-1,r-1)})) = \{I_{n_j-1}(\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \cdots, b_{m_j}^{(j)}) : 1 \leq b_1^{(j)}, \cdots, b_{m_j}^{(j)} \leq M\}$$

and let

$$\mathcal{F}_j = \bigcup_{I_{n_{j-1}+(r-1)}\in\mathcal{E}_{j-1,r-1}} \mathcal{F}_j(I_{n_{j-1}+(r-1)}(\nu^{(j-1,r-1)})).$$

Following the same process as for **Level 1**, for each $I_{n_j-1}(w^{(j)}) \in \mathcal{F}_j$ define the collection of sub-cylinders:

$$\mathcal{E}_{j,0}(w^{(j)}) := \{I_{n_j}(\nu^{(j,0)}) := I_{n_j}(w^{(j)}, a_{n_j}) : \\ \gamma_0^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \leq a_{n_j} < 2\gamma_0^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\}$$

where $z_j \in I_{n_j-(n_{j-1}+(r-1))-1}(\nu^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \cdots, b_{m_j}^{(j)})$.

Next for each $I_{n_j}(\nu^{(j,0)}) \in \mathcal{E}_{j,0}(w^{(j)})$, define

$$\mathcal{E}_{j,1}(\nu^{(j,0)}) := \{I_{n_j+1}(\nu^{(j,1)}) := I_{n_j+1}(\nu^{(j,0)}, a_{n_j+1}) : \\ \gamma_1^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \leq a_{n_j+1} < 2\gamma_1^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\}.$$

Similarly for each $I_{n_j+i-1}(\nu^{(j,i-1)}) \in \mathcal{E}_{j,i-1}(\nu^{(j,i-2)})$, with $2 \leq i \leq r-2$ collect a family of sub-cylinders of order n_{j+i}:

$$\mathcal{E}_{j,i}(\nu^{(j,i-1)}) := \{I_{n_j+i}(\nu^{(j,i)}) = I_{n_j+i}(\nu^{(j,i-1)}, a_{n_j+i}) : \\ \gamma_i^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \leq a_{n_1+i} < 2\gamma_i^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\}.$$

Continuing in this way for each $I_{n_j+r-2}(\nu^{(j,r-2)}) \in \mathcal{E}_{j,r-2}(\nu^{(j,r-3)})$ we define

$$\mathcal{E}_{j,r-1}(\nu^{(j,r-2)}) := \{I_{n_j+(r-1)}(\nu^{(j,r-1)}) := I_{n_j+(r-1)}(\nu^{(j,r-2)}, a_{n_j+(r-1)})) : \\ \left(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\right)^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \\ \leq a_{n_j+(r-1)} < 2\left(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\right)^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\}.$$

Then the jth level '$\mathcal{E}_{j,r-1}$' of the Cantor set $\mathcal{E}_\infty$ is defined as

$$\mathcal{E}_{j,r-1} = \{I_{n_j+(r-1)}(\nu^{(j,r-1)}) \in \mathcal{E}_{j,r-1}(\nu^{(j,r-2)}) : \\ I_{n_j+i}(\nu^{(j,i)}) \in \mathcal{E}_{j,i}(\nu^{(j,i-1)}) \text{ for } 1 \leq i \leq r-2; \\ I_{n_j}(\nu^{(j,0)}) \in \mathcal{E}_{j,0}(w^{(j)}); I_{n_j-1}(w^{(j)}) \in \mathcal{F}_j\}.$$

Then the Cantor set is defined as

$$\mathcal{E}_\infty := \bigcap_{j=1}^{\infty} \bigcup_{I_{n_j+(r-1)}(\nu^{(j,r-1)})\in\mathcal{E}_{j,r-1}} I_{n_j+(r-1)}(\nu^{(j,r-1)}).$$

By the same arguments as discussed in **Level 1**, that is, by the continuity of τ and since z_j and x share common prefixes up to t_jth partial quotients,

$$\lim_{j\to\infty} \tau(z_j) = \tau(x) \text{ and } \lim_{j\to\infty} \frac{S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}{S_{n_j-1}h(x)} = 1. \tag{6.11}$$

Therefore $\mathcal{E}_\infty$ is contained in $\mathcal{R}_r(\tau; h)$. As one can easily check that for showing $\mathcal{E}_\infty \subset \mathcal{R}_r(\tau; h)$ it is sufficient to show that (6.11) holds.

In order to better understand the structure of $\mathcal{E}_\infty$ we will utilize the idea of symbolic space. If $[\nu_1, \nu_2, \cdots, \nu_n, \cdots]$ is a continued fraction expansion of a point $x \in \mathcal{E}_\infty$ then call the sequence $(\nu_1, \nu_2, \cdots, \nu_n, \cdots)$ an *admissible sequence* and for any $n \geq 1$ call the finite truncation $\nu := (\nu_1, \nu_2, \cdots, \nu_n)$ an *admissible block*. If ν is an admissible block only than $I_n(\nu) \cap \mathcal{E}_\infty \neq \emptyset$, and such basic cylinders $I_n(\nu)$ are known as *admissible cylinders*. For any $n \geq 1$, denote by '$\mathcal{D}_n$' the set of strings defined as

$$\mathcal{D}_n = \{(\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{N}^n : \nu = (\nu_1, \nu_2, \cdots, \nu_n) \text{ is an admissible block } \}.$$

Next we will recursively define $\mathcal{D}_n$, according to the limitations on the partial quotients defined for different cases in the construction of $\mathcal{E}_\infty$.

Write $l_1 = n_1 - (n_0 + (r-1)) - 1 = t_1 + m_1$.

(1a) When $1 \leq n \leq (n_0 + (r-1))$,

$$\mathcal{D}_n = \{\nu^{(0,r-1)} = (a_1(z_0), a_2(z_0), \cdots, a_{n_0+(r-1)}(z_0))\}.$$

(1b) When $(n_0 + (r-1)) < n \leq (n_0 + (r-1)) + t_1$,

$$\mathcal{D}_n = \{(\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{n-(n_0+(r-1))})\}.$$

(1c) When $(n_0 + (r-1)) + t_1 < n < n_1$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1)}, \ldots, \nu_n) :\\ 1 \leq \nu_u \leq M,\ (n_0 + (r-1)) + t_1 < u \leq n\}.\end{aligned}$$

(1d) When $n = n_1$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1)}, \ldots, \nu_{n_1-1}, \nu_{n_1}) :\\ \gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \leq \nu_{n_1} < 2\gamma_0^{\tau(z_1)S_{l_1}h(z_1)}\\ \text{and } 1 \leq \nu_u \leq M \text{ for } (n_0 + (r-1)) + t_1 < u < n_1\},\end{aligned}$$

where $z_1 \in I_{l_1(\nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1}, \cdots, \nu_{n_1-1})}$.

(1e) When $n = n_1 + i$ where $1 \leq i \leq r-2$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1)}, \ldots, \nu_{n_1+i-1}, \nu_{n_1+i}) :\\ \gamma_i^{\tau(z_1)S_{l_1}h(z_1)} \leq \nu_{n_1+i} < 2\gamma_i^{\tau(z_1)S_{l_1}h(z_1)} \text{ where } 1 \leq i \leq r-2,\\ \gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \leq \nu_{n_1} < 2\gamma_0^{\tau(z_1)S_{l_1}h(z_1)}\\ \text{and } 1 \leq \nu_u \leq M \text{ for } (n_0 + (r-1)) + t_1 < u < n_1\}.\end{aligned}$$

(1f) When $n = n_1 + (r-1)$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1)}, \ldots, \nu_{n_1-1}, \nu_{n_1}) : \\ \left(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\right)^{\tau(z_1)S_{l_1}h(z_1)} \leq \nu_{n_1+(r-1)} < 2\left(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\right)^{\tau(z_1)S_{l_1}h(z_1)}, \\ \gamma_i^{\tau(z_1)S_{l_1}h(z_1)} \leq \nu_{n_1+i} < 2\gamma_i^{\tau(z_1)S_{l_1}h(z_1)} \text{ where } 1 \leq i \leq r-2, \\ \gamma_0^{\tau(z_1)S_{l_1}f(z_1)} \leq \nu_{n_1} < 2\gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \\ \text{and } 1 \leq \nu_u \leq M \text{ for } (n_0+(r-1))+t_1 < u < n_1\}.\end{aligned}$$

Next to define $\mathcal{D}_n$ inductively, suppose that $\mathcal{D}_{n_{j-1}+(r-1)}$ has been defined. For each $j \geq 1$, write $l_j = n_j - (n_{j-1} + (r-1)) - 1 = t_j + m_j$.

(2a) When $(n_{j-1} + (r-1)) < n \leq (n_{j-1} + (r-1)) + t_j$,

$$\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{n-(n_{j-1}+(r-1))})\}.$$

(2b) When $(n_{j-1} + (r-1)) + t_j < n < n_j$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1}, \ldots, \nu_n) : \\ \nu^{(j-1,r-1)} \in \mathcal{D}_{n_{j-1}+(r-1)}, \\ 1 \leq \nu_u \leq M,\ (n_{j-1}+(r-1))+t_j < u \leq n\}.\end{aligned}$$

(2c) When $n = n_j$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_1+1)}, \ldots, \nu_{n_j-1}, \nu_{n_j}) : \\ \gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \\ \text{and } 1 \leq \nu_u \leq M \text{ for } (n_{j-1}+(r-1))+t_j < u < n_j\}.\end{aligned}$$

where $z_j \in I_{l_j}(\nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1}, \cdots, \nu_{n_j-1})$.

(2d) When $n = n_j + i$ where $1 \leq i \leq r-2$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1)}, \ldots, \nu_{n_j+i-1}, \nu_{n_j+i}) : \\ \gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j+i} < 2\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \text{ where } 1 \leq i \leq r-2, \\ \gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \\ \text{and } 1 \leq \nu_u \leq M \text{ for } (n_{j-1}+(r-1))+t_j < u < n_j\}.\end{aligned}$$

(2e) When $n = n_j + (r-1)$,

$$\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1)}, \ldots, \nu_{n_j-1}, \nu_{n_j}) :$$

$$\left(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j+(r-1)} < 2\left(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)},$$
$$\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j+i} < 2\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \text{ where } 1 \leq i \leq r-2,$$
$$\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}$$
$$\text{and } 1 \leq \nu_u \leq M, \text{ for } (n_{j-1}+(r-1))+t_j < u < n_j\}.$$

Fundamental cylinders

For each $\nu = (\nu_1, \cdots, \nu_n) \in \mathcal{D}_n$, we define a fundamental cylinder J_n as the union of basic cylinders I_n, having a non empty intersection with $\mathcal{E}_\infty$.

(3a) For $(n_{j-1}+(r-1))+t_j < n < n_j+1$, define

$$J_n(\nu) = \bigcup_{1\leq\nu_{n+1}\leq M} I_{n+1}(\nu_1,\ldots,\nu_n,\nu_{n+1}). \tag{6.12}$$

(3b) For $n = n_j - 1$, define

$$J_{n_j-1}(\nu) = \bigcup_{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}} I_{n_j}(\nu_1,\ldots,\nu_{n_j-1},\nu_{n_j}), \tag{6.13}$$

where $z_j \in I_{l_j}(\nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1}, \cdots, \nu_{n_j-1})$.

(3c) For $n = n_j + i - 1$ with $1 \leq i \leq r-2$, define

$$J_{n_j+i-1}(\nu) = \bigcup_{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j+i} \leq 2\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}} I_{n_j+i}(\nu_1,\ldots,\nu_{n_j+i-1},\nu_{n_j+i}). \tag{6.14}$$

(3d) For $n = n_j + (r-2)$, define

$$J_{n_j+(r-2)}(\nu) = \bigcup_{(\frac{e}{\gamma_0\cdots\gamma_{r-2}})^{\tau(z_j)S_{l_j}h(z_j)} \leq \nu_{n_j+(r-1)} \leq 2(\frac{e}{\gamma_0\cdots\gamma_{r-2}})^{\tau(z_j)S_{l_j}h(z_j)}} I_{n_j+(r-1)}(w), \tag{6.15}$$

where $w = (\nu_1, \ldots, \nu_{n_j+(r-1)})$.

(3e) For $n_j + (r-1) \leq n \leq (n_j+(r-1)) + t_{j+1}$, then by construction of $\mathcal{E}_\infty$ we have

$$J_n(\nu) = I_{n_j+(r-1)+t_{j+1}}(\nu^{(j,r-1)}, \nu^{(j,r-1)}|_{t_{j+1}}). \tag{6.16}$$

Clearly,

$$\mathcal{E}_\infty = \bigcap_{n=1}^{\infty} \bigcup_{\nu\in D_n} J_n(\nu).$$

Lengths of fundamental cylinders

In the following subsection we will estimate the lengths of the fundamental cylinders for different cases discussed above.

Let $[\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \cdots, b_{m_j}^{(j)}, a_{n_j}, \cdots, a_{n_j+(r-1)}, \cdots]$ be a continued fraction representation for any point $x \in \mathcal{E}_\infty$.

I. If $n = (n_j + (r-1)) + t_{j+1}$, then by using (2.5)

$$\begin{aligned} &q_{(n_j+(r-1))+t_{j+1}}(\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \cdots, b_{m_j}^{(j)}, a_{n_j}, \cdots, a_{n_j+(r-1)}, \nu^{(j,r-1)}|_{t_{j+1}}) \\ &\quad \le 2^{3r+2} q_{(n_{j-1}+(r-1))+t_j}(\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}) \cdot q_{m_j}(b_1^{(j)}, \cdots, b_{m_j}^{(j)}) \cdot e^{\tau(z_j)S_{l_j}h(z_j)} \\ &\qquad \cdot q_{t_{j+1}}(\nu^{(j,r-1)}|_{t_{j+1}}). \end{aligned}$$

Next by using the fact that $q_{m_j} \ge 2^{\frac{m_j-1}{2}}$ and by the choice of m_j,

$$\begin{aligned} q_{(n_j+(r-1))+t_{j+1}}(x) &\le q_{(n_{j-1}+(r-1))+t_j}(x) \cdot (q_{l_j}(z_j) e^{\tau(z_j)S_{l_j}h(z_j)})^{1+\epsilon} \\ &\le \prod_{k=1}^{j} (q_{l_k}(z_k) e^{\tau(z_k)S_{l_k}h(z_k)})^{1+\epsilon}, \end{aligned} \tag{6.17}$$

where $z_k \in I_{l_k}(\nu^{(k-1,r-1)}|_{t_k}, b_1^{(k)}, \cdots, b_{m_k}^{(k)})$ for all $1 \le k \le j$.

II. If $(n_j + (r-1)) \le n < (n_j + (r-1)) + t_{j+1}$, then

$$q_n(x) \le q_{(n_j+(r-1))+t_{j+1}}(x) \le \prod_{k=1}^{j} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{1+\epsilon}.$$

III. If $(n_{j-1} + (r-1)) + t_j \le n \le n_j - 1$, and if we represent $n - (n_{j-1} + (r-1)) - t_j$ by l, then

$$\begin{aligned} q_n(x) &\le 2 q_{(n_{j-1}+(r-1))+t_j}(x) \cdot q_l(b_1^{(j)}, \cdots, b_l^{(j)}) \\ &\le \prod_{k=1}^{j-1} (q_{l_k}(z_k) e^{\tau(z_k)S_{l_k}h(z_k)})^{1+\epsilon} \cdot q_l(b_1^{(j)}, \cdots, b_l^{(j)}). \end{aligned}$$

Now we calculate the lengths of fundamental cylinders for different cases as defined above (6.12)–(6.16).

I. If $(n_{j-1} + (r-1)) \le n \le (n_{j-1} + (r-1)) + t_j$, then by using (2.2), (6.16) and (6.17)

$$\begin{aligned} |J_n(x)| &= \left|I_{(n_{j-1}+(r-1))+t_j}(x)\right| \ge \frac{1}{2 q^2_{(n_{j-1}+(r-1))+t_j}(x)} \\ &\ge \frac{1}{2} \prod_{k=1}^{j-1} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}. \end{aligned}$$

II. If $(n_{j-1} + (r-1)) + t_j < n < n_j - 1$ and $l = n - (n_{j-1} + (r-1)) - t_j - 1$, then from (2.2) and (6.12)

$$|J_n(x)| \ge \frac{1}{6 q_n^2(x)} \ge \frac{1}{6} \prod_{k=1}^{j} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)} \cdot q_l^{-2}(b_1^{(j)}, \cdots, b_l^{(j)}).$$

III. If $n = n_j - 1$ then by using (6.13) and following the similar steps as for **I**

$$|J_{n_j-1}(x)| \geq \frac{1}{6\nu_{n_j}(x)q_{n_j-1}^2(x)} \geq \frac{1}{6\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{n_j-1}^2(x)}$$
$$\geq \frac{1}{24\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(x)} \cdot \prod_{k=1}^{j-1}(q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}.$$

IV. If $n = n_j + i - 1$ where $1 \leq i \leq r-2$ then from (6.14) and following the similar steps as for **I,**

$$|J_{n_j+i-1}(x)| \geq \frac{1}{6\nu_{n_j+i}(x)q_{n_j+i-1}^2(x)} \geq \frac{1}{6\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}q_{n_j+i-1}^2(x)}$$
$$\geq \frac{1}{6\cdot 4^i\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}(\gamma_0\cdots\gamma_{i-1})^{2\tau(z_j)S_{l_j}h(z_j)}q_{n_j-1}^2(x)}$$
$$\geq \frac{1}{6\cdot 4^i\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}(\gamma_0\cdots\gamma_{i-1})^{2\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}$$
$$\cdot \prod_{k=1}^{j-1}(q_{l_k}(z_k)e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}.$$

V. If $n = n_j + (r-2)$ then from (6.15)

$$|J_{n_j+(r-2)}(x)| \geq \frac{1}{6\nu_{n_j+(r-1)}(x)q_{n_j+(r-2)}^2(x)}$$
$$\geq \frac{1}{6\cdot 4^{r-1}(e\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}q_{n_j-1}^2(x)}$$
$$\geq \frac{1}{6\cdot 4^{r}(e\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)} \cdot \prod_{k=1}^{j-1}(q_{l_k}(z_k)e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}.$$

Supporting measure

In this subsection we will define a probability measure supported on the set $\mathcal{E}_\infty$.

Define $s_j := s^{(r)}_{(t_j,m_j),M}$ to be the solution of

$$\sum_{a_1=\nu_1^{(j-1,r-1)},\cdots,a_{t_j}=\nu_{t_j}^{(j-1,r-1)},1\leq b_1^{(j)},\cdots,b_{m_j}^{(j)}\leq M} \frac{1}{e^{g_r(s)\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^{2s}(z_j)} = 1$$

where $z_j \in I_{l_j}(\nu^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \cdots, b_{m_j}^{(j)})$. Consequently from (6.9),

$$\sum_{a_1=\nu_1^{(j-1,r-1)},\cdots,a_{t_j}=\nu_{t_j}^{(j-1,r-1)},1\leq b_1^{(j)},\cdots,b_{m_j}^{(j)}\leq M} \left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^s = 1 \tag{6.18}$$

where $z_j \in I_{l_j}(\nu^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \cdots, b_{m_j}^{(j)})$.

Equality (6.18) induces a measure μ on basic cylinder of order t_j+m_j if we consider

$$\mu(I_{n_j+t_j}(a_1,\cdots,a_{t_j},b_1^{(j)},\cdots,b_{m_j}^{(j)}))=\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{s_j},$$

for each $a_1=\nu_1^{(j-1,r-1)},\cdots,a_{t_j}=\nu_{t_j}^{(j-1,r-1)},1\le b_1^{(j)},\cdots,b_{m_j}^{(j)}\le M.$

We will start by assuming that the measure of $I_{n_{j-1}+(r-1)}(x)\in\mathcal{E}_\infty$ has been defined as

$$\mu\left(I_{n_{j-1}+(r-1)}(x)\right)=\prod_{k=1}^{j-1}\left(\left(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\right),$$

where $z_k\in I_{l_k}(\nu^{(k-1,r-1)}|_{t_k},b_1^{(k)},\cdots,b_{m_k}^{(k)})$ for all $1\le k\le j-1$.

Case 1: $n_{j-1}+(r-1)<n\le n_{j-1}+(r-1)+t_j$. As the basic cylinder of order $n_{j-1}+(r-1)$ contains only one sub-cylinder of order n with a non-empty intersection with $\mathcal{E}_\infty$, therefore

$$\mu(I_n(x))=\mu\left(I_{n_{j-1}+(r-1)}(x)\right).$$

Case 2: $n=n_j-1$. Let

$$\mu(I_{n_j-1}(x))=\mu\left(I_{n_{j-1}+(r-1)}(x)\right)\cdot\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{s_j}.$$

Next we will uniformly distribute the measure of $I_{n_j-1}(x)$ on its sub-cylinders.

Case 3: $n=n_j+i-1$, where $1\le i\le r-1$.

$$\begin{aligned}\mu(I_{n_j+i-1}(x))&=\mu\left(I_{n_j+i-2}(x)\right)\cdot\frac{1}{\gamma_{i-1}^{\tau(z_j)S_{l_j}h(z_j)}}\\&=\mu\left(I_{n_j-1}(x)\right)\cdot\frac{1}{(\gamma_0\cdots\gamma_{i-1})^{\tau(z_j)S_{l_j}h(z_j)}}.\end{aligned}$$

Case 4: $n=n_j+(r-1)$.

$$\begin{aligned}\mu(I_{n_j+r-1}(x))&=(\frac{\gamma_0\cdots\gamma_{r-2}}{e})^{\tau(z_j)S_{l_j}h(z_j)}\mu\left(I_{n_j+(r-2)}(x)\right)\\&=(\frac{\gamma_0\cdots\gamma_{r-2}}{e})^{\tau(z_j)S_{l_j}h(z_j)}\frac{1}{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}\mu\left(I_{n_j+(r-2)}(x)\right)\\&=\frac{1}{e^{\tau(z_j)S_{l_j}h(z_j)}}\mu\left(I_{n_j-1}(x)\right).\end{aligned}$$

The measure of other basic cylinders of order less than n_j-1 is followed by the consistency property that a measure should satisfy.

For any $n_{j-1}+(r-1)+t_j<n\le n_j-1$, let

$$\mu(I_n(x))=\sum_{I_{n_j-1(x)}\subset I_n(x)}\mu\left(I_{n_j-1}(x)\right).$$

The Hölder exponent of the measure μ

In this part we will compare the measure of fundamental cylinders with their lengths.
Case 1: $n = n_j - 1$.

$$\begin{aligned}
\mu\left(J_{n_j-1}(x)\right) &= \prod_{k=1}^{j-1}\left(\left(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\right)\cdot\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{s_j}\\
&\le \prod_{k=1}^{j-1}\left(\frac{1}{\gamma_0^{s_k\tau(z_k)S_{l_k}h(z_k)}e^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^{2s_k}(z_k)}\right)\cdot\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{s_M^{(r)}-3\epsilon}\\
&\le \prod_{k=1}^{j-1}\left(\frac{1}{e^{2s_k\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^{2s_k}(z_k)}\right)\cdot\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{s_M^{(r)}-3\epsilon} \qquad (6.19)\\
&\le \left(\prod_{k=1}^{j-1}\left(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{1+\epsilon}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\cdot\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\\
&\le 24|J_{n_j-1}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}.
\end{aligned}$$

From (6.4), we observe that $|s_j - s_M^{(r)}| \le 3\epsilon$ which further implies that $s_M^{(r)} - 3\epsilon \le s_j$. In (6.19), we have used the fact that since $1 \le \gamma_0\cdots\gamma_{r-2} \le e$ and $\frac{e}{\gamma_0\cdots\gamma_{r-2}} \ge \left(\frac{e}{\gamma_0\cdots\gamma_{r-2}}\right)^s$ for any $0 < s < 1$, we have $e\gamma_0^s \ge e^{2s}$. Therefore it is also true for s_k.
Case 2: $n = n_j + i - 1$, where $1 \le i \le r-2$. As we know that

$$\begin{aligned}
\mu(J_{n_j+i-1}(x)) &= \mu\left(I_{n_j-1}(x)\right)\cdot\frac{1}{(\gamma_0\cdots\gamma_{i-1})^{\tau(z_j)S_{l_j}h(z_j)}}\\
&\le \left(\prod_{k=1}^{j-1}\left(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{1+\epsilon}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\cdot\left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\\
&\quad\cdot\frac{1}{(\gamma_0(\gamma_1\cdots\gamma_{i-1})^2\gamma_i)^{\tau(z_j)S_{l_j}h(z_j)}} \qquad (6.20)\\
&\le \left(\prod_{k=1}^{j-1}\left(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{1+\epsilon}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\\
&\le 6.4^{i+1}|J_{n_j+i-1}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\\
&\le 6\cdot 4^{r-3}|J_{n_j+i-1}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}.
\end{aligned}$$

We have obtained (6.20) by using the fact the $\frac{1}{\gamma_0\gamma_1\cdots\gamma_{i-1}} \le \left(\frac{1}{\gamma_0(\gamma_1\cdots\gamma_{i-1})^2\gamma_i}\right)^s$ for any $0 < s < 1$.

Case 3: $n = n_j + r - 2$.

$$\begin{aligned}
\mu(J_{n_j+r-2}(x)) &= \mu\left(J_{n_j-1}(x)\right) \cdot \frac{1}{(\gamma_0 \cdots \gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}} \\
&\leq \left(\prod_{k=1}^{j-1}\left(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{1+\epsilon}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\
&\cdot \left(\frac{1}{(e\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\
&\leq 6.4^r |J_{n_j+r-2}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}.
\end{aligned}$$

Case 4: $n_j + (r-1) \leq n \leq n_j + t_{j+1}$.

$$\begin{aligned}
\mu(J_n(x)) &= \prod_{k=1}^{j-1}\left(\left(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\right) \\
&\cdot \left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\right)^{s_j}\frac{1}{e^{\tau(z_j)S_{l_j}h(z_j)}} \\
&= \prod_{k=1}^{j}\left(\left(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\right) \\
&\leq \prod_{k=1}^{j}\left(\frac{1}{e_0^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{s_k} \\
&\leq \prod_{k=1}^{j}\left(\frac{1}{e_0^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{s_M^{(r)}-3\epsilon} \\
&\leq \left(\prod_{k=1}^{j}\left(\frac{1}{e_0^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\right)^{1+\epsilon}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \qquad \leq 2|J_n(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}.
\end{aligned}$$

Gap estimation

Let $x \in \mathcal{E}_\infty$. In this section we estimate the gap between $J_n(x)$ and its adjacent fundamental cylinder $J_n(x')$ of the same order n. Assume that $a_i(x) = a_i(x')$ for all $1 \leq i < n$. These gaps are helpful for estimating the measure on general balls. Also as $J_n(x)$ and $J_n(x')$ are adjacent, we have $|a_n(x) - a_n(x')| = 1$.

Let the left and the right gap between $J_n(x)$ and its adjacent fundamental cylinder at each side be represented by $g_n^L(x)$ and $g_n^R(x)$ respectively.

Denote by $g_n^{L,R}(x)$ the minimum distance between $J_n(x)$ and its adjacent cylinder of the same order n, that is,

$$g_n^{L,R}(x) = \min\{g_n^L(x), g_n^R(x)\}.$$

Without loss of generality we assume that n is even and estimate $g_n^R(x)$ only, since if n is odd then for $g_n^L(x)$ we can carry out the estimation in almost the same way.

Gap I. When $(n_{j-1} + (r-1)) + t_j < n < n_j - 1$, for all $j \geq 1$,

$$\begin{aligned} g_n^R(x) &\geq \sum_{a_{n+1}>M} |I_{n+1}(a_1, a_2, \ldots, a_{n-1}, a_n + 1, a_{n+1})| \\ &= \frac{(M+1)(p_n + p_{n-1}) + p_{n-1}}{(M+1)(q_n + q_{n-1}) + q_{n-1}} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \\ &= \frac{1}{((M+1)(q_n + q_{n-1}) + q_{n-1})(q_n + q_{n-1})} \\ &\geq \frac{1}{3Mq_n^2} \geq \frac{1}{3M}|I_n(x)|. \end{aligned}$$

Gap II. When $n = n_j + i - 1$ where $0 \leq i \leq r-2$,

$$\begin{aligned} g_n^R(x) &\geq \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} p_n + p_{n-1}}{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n + q_{n-1}} \\ &= \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} - 1}{\left(\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n + q_{n-1}\right)(q_n + q_{n-1})} \geq \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} - 1}{4\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n^2} \\ &\geq \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}}{8\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n^2} \geq \frac{1}{8}|I_n(x)|. \end{aligned}$$

Gap III. When $n = n_j + r - 2$,

$$\begin{aligned} g_n^R(x) &\geq \frac{\left(\frac{e}{\gamma_0 \cdots \gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)} - 1}{\left(\left(\frac{e}{\gamma_0 \cdots \gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)} q_n + q_{n-1}\right)(q_n + q_{n-1})} \\ &\geq \frac{1}{8q_n^2} \geq \frac{1}{8}|I_n(x)|. \end{aligned}$$

Gap IV. If $(n_j + (r-1)) \leq n \leq (n_j + (r-1)) + t_{j+1}$ then note that $J_n(x)$ is a small part of $I_{n_j+(r-1)}(x)$ as $I_{(n_j+(r-1))+t_{j+1}}(x) \subset I_{(n_j+(r-1))+2}(x)$. Therefore, the right gap is larger than the distance between the right endpoints of $J_n(x)$ and that of $I_{n_j+(r-1)}(x)$.

$$g_n^R(x) \geq \left|I_{(n_j+(r-1))+2}(x)\right| \geq \frac{1}{2q^2_{(n_j+(r-1))+2}(x)} \geq \frac{1}{32a_1^2a_2^2q^2_{n_j+(r-1)}(x)}$$

$$\begin{aligned} &\geq \frac{1}{32a_1^2a_2^2q^2_{(n_j+(r-1))+t_j}(x)} \geq \frac{1}{32a_1^2a_2^2}|I_{(n_j+(r-1))+t_j}(x)| \\ &= \frac{1}{32a_1^2a_2^2}|J_{(n_j+(r-1))+t_j}(x)|, \end{aligned}$$

where a_1 represents $a_{(n_j+(r-1))+1}(x)$ and a_2 represents $a_{(n_j+(r-1))+2}(x)$.

The measure μ on general ball $B(x,d)$

We now estimate the measure μ on any ball $B(x,d)$ with radius d and centred at x. Fix $x \in \mathcal{E}_\infty$. There exists a unique sequence $(\nu_1, \nu_2, \cdots \nu_n, \cdots)$ such that for each $n \geq 1$, $x \in J_n(\nu_1, \cdots, \nu_n)$ where $(\nu_1 \cdots, \nu_n) \in \mathcal{D}_n$ and

$$g^R_{n+1}(x) \leq d < g^R_n(x).$$

Clearly, $B(x,d)$ can intersect only one fundamental cylinder of order n, i.e., $J_n(\nu_1, \ldots, \nu_n)$.
Case I: $(n_{j-1} + (r-1)) + t_j < n < n_j - 1$, for all $j \geq 1$ or $n = n_j + t_{j+1}$. Since in this case $1 \leq a_n(x) \leq M$ and $|J_n(x)| \leq \frac{1}{q_n^2}$, thus we have

$$\begin{aligned} \mu(B(x,d)) &\leq \mu(J_n(x)) \leq c|J_n(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\ &\leq c\left(\frac{1}{q_n^2}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\ &\leq c4M^2\left(\frac{1}{q_{n+1}^2}\right)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\ &\leq c8M^2|I_{n+1}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\ &\leq c48M^3g^R_{n+1}(x)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \\ &\leq Cd^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}, \text{ where } C = cc_0^3 \text{ are arbitrary constants.} \end{aligned}$$

Case II: $n = n_j + i - 1$ where $0 \leq i \leq r-2$. Since

$$\frac{1}{8\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)}q^2_{n_j+i-1}(x)} \leq |I_{n_j+i}(x)| \leq 8g^R_{n_j+i}(x) \leq 8d$$

implies

$$1 \leq 64d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)}q^2_{n_j+i-1}(x),$$

the number of fundamental cylinders of order n_j+i contained in $J_{n_j+i-1}(x)$ that the ball $B(x,d)$ intersects is at most

$$\frac{2d}{|I_{n_j+i}(x)|} + 2 \leq 16d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)}q^2_{n_j+i-1}(x) + 2^7d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)}$$

$$= c_0 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x).$$

Therefore,

$$\begin{aligned}
\mu(B(x,d)) &\le \min\Big\{\mu(J_{n_j+i-1}(x)), c_0 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\mu(J_{n_j+i}(x))\Big\}\\
&\le \mu(J_{n_j+i-1}(x)) \min\Big\{1, c_0 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\frac{1}{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}}\Big\}\\
&\le 6\cdot 4^{i+1}|J_{n_j+i-1}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \min\Big\{1, c_0 d\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\Big\}\\
&\le c\Big(\frac{1}{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)}\Big)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \Big(c_0 d\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\Big)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\\
&\le cc_0 d^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}.
\end{aligned}$$

Here we have used the fact that $\min\{a,b\} \le a^{1-s}b^s$ for any $a,b>0$ and $0\le s\le 1$.
Case III: $n = n_j + r - 2$. As

$$\frac{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}}{8e^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+r-2}(x)} \le |I_{n_j+r-1}(x)| \le 8g^R_{n_j+r-1}(x) \le 8d,$$

the number of fundamental cylinders of order $n_j + r - 1$ contained in $J_{n_j+r-2}(x)$ that the ball $B(x,d)$ intersects is at most

$$\frac{2d}{|I_{n_j+r-1}(x)|} + 2 \le c_0 d\frac{e^{2\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-1}(x).$$

Therefore,

$$\begin{aligned}
&\mu(B(x,d))\\
&\le \min\Big\{\mu(J_{n_j+r-2}(x)), c_0 d\frac{e^{2\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\mu(J_{n_j+r-1}(x))\Big\}\\
&\le \mu(J_{n_j+r-2}(x)) \min\Big\{1, c_0 d\frac{e^{2\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\frac{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}{e^{\tau(z_j)S_{l_j}h(z_j)}}\Big\}\\
&\le 6.4^r|J_{n_j+r-2}(x)|^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} \min\Big\{1, c_0 d\frac{e^{\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\Big\}\\
&\le c\Big(\frac{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}{e^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+r-2}}\Big)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}} .\Big(c_0 d\frac{e^{\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\Big)^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}\\
&\le cc_0 d^{\frac{s_M^{(r)}-3\epsilon}{1+\epsilon}}.
\end{aligned}$$

By combining all the above cases and using the mass distribution principle, we conclude that

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau;h) \ge \dim_{\mathrm{H}} \mathcal{E}_\infty \ge s_0 = \frac{s_M^{(r)} - 3\epsilon}{1+\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, as $\epsilon \to 0$ we have $s_0 \to s_M^{(r)}$. Further, letting $M \to \infty$, we obtain

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h) \geq s_{\mathbb{N}}^{(r)}.$$

This completes the proof for the lower bound and thus of Theorem 6.1.1.

Chapter 7

A survey of advances in uniform Diophantine approximation and open problems

We will conclude the thesis by this short survey which summarises the findings of this thesis and list some recent results as well as some open problems arising in the uniform approximation theory concerned with the metrical theory of the sets of Dirichlet non-improvable numbers.

7.1 Some recent developments to uniform Diophantine approximation theory

In the one-dimensional Diophantine approximation, we have notice that by using the theory of continued fractions Khintchine and Jarník theorems are concerned with the growth of the large partial quotients while the improvability of Dirichlet's Theorem is concerned with the growth of the product of consecutive partial quotients.

Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t_0 \geq 1$ fixed and $t\psi(t) < 1$ for all $t \geq t_0$. Recall that if a real number $x \in D(\psi)$, where

$$D(\psi) = \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\, N \text{ such that the system } |qx - p| < \psi(t), |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\},$$

then x is known as ψ-Dirichlet improvable and if it belongs to the complementary set $D(\psi)^c$ then it is called ψ-Dirichlet non-improvable. From the discussion of Chapter 1, it is obvious that a lot has been done to strengthen Corollary 1.1.3 rather than Theorem 1.1.2. To this end, a natural question is to investigate the set $D(\psi)$.

As discussed in Chapter 3, Kleinbock–Wadleigh [34] provided the Lebesgue measure criterion for this set. Recall that the auxiliary function Ψ is defined as

$$\Psi(t) = \frac{1}{1 - t\psi(t)} - 1,$$

with $t\psi(t) < 1$ for all $t > t_0$.

Theorem 7.1.1 (Kleinbock–Wadleigh, [34]) *Let ψ and Ψ be as defined above. Then*

$$\lambda(D^c(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} < \infty; \\ 1 & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} = \infty. \end{cases} \tag{7.1}$$

Whereas the Hausdorff measure version for $D^c(\psi)$ has been established in [27].

Theorem 7.1.2 (Hussain–Kleinbock–Wadleigh–Wang, [27]) *Let Ψ and ψ be functions as defined above. Then for any $s \in [0,1)$*

$$\mathcal{H}^s(D(\psi)^c) = \begin{cases} 0 & \text{if} \quad \sum_t t\left(\frac{1}{t^2\Psi(t)}\right)^s < \infty; \\ \infty & \text{if} \quad \sum_t t\left(\frac{1}{t^2\Psi(t)}\right)^s = \infty. \end{cases} \tag{7.2}$$

Consequently,

$$\dim_{\mathrm{H}} D(\psi)^c = \frac{2}{2+\tau}; \quad \tau = \liminf_{t\to\infty} \frac{\log \Psi(t)}{\log t}$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension.

The condition $s < 1$ is necessary, $\mathcal{H}^1$ is the Lesbesgue measure which is the scope of zero-one law by Kleinbock–Wadleigh [34]. The summability criterion that appears in (7.1) does not agree with the one in (7.2). Indeed when $s = 1$ the summand in equation (7.2) differs from that in equation (7.1) by a factor of $\log \Psi(t)$.

A natural generalisation of the s-dimensional Hausdorff measure is the f-dimensional Hausdorff measure $\mathcal{H}^f$ where f is a *dimension function*, that is an increasing, continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \to 0$ as $r \to 0$. We need to impose an additional technical condition on f: say that a dimension function f is *essentially sub-linear* if there exists

$$B > 1 \text{ such that } \limsup_{x\to 0} \frac{f(Bx)}{f(x)} < B. \tag{7.3}$$

The above condition does not hold for $f(x) = x$ or $f(x) = x\log(1/x)$. However it is clearly satisfied for the dimension functions $f(x) = x^s$ when $s \in [0,1)$. Further, we remark that the essentially sub-linear condition is equivalent to the doubling condition but with exponent $\alpha < 1$.(A function f is called *doubling* with exponent α if $f(cx) \ll c^\alpha f(x)$ for all x and all $c > 1$).

The following theorem readily implies Theorem 7.1.2.

Theorem 7.1.3 (Hussain–Kleinbock–Wadleigh–Wang, [27]) *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t, and let f be an essentially sub-linear dimension function. Then*

$$\mathcal{H}^f\left(D(\psi)^c\right) = \begin{cases} 0 & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

Naturally one would like to investigate the Hausdorff f-measure of sets $D(\psi)^c$ for a large class of *non-essentially sub-linear* dimension function. The only restriction that we have on the dimension functions, in addition to negating (7.3), is that the dimension functions are not the identity functions.

Returning to the theory of continued fractions, as noticed in Chapter 1, partial quotients reveal how rapidly a real number can be approximated by rationals. Thus it motivates us to express the elements of $D(\psi)$ (or $D(\psi)^c$) in terms of entries of continued fraction expansion. Kleinbock–Wadleigh [34] provided a characterisation of the ψ-Dirichlet improvable number x in terms of growth of partial quotients in the continued fraction of x.

This observation leads to the following characterisation of Dirichlet's improvable numbers.

Lemma 7.1.4 (Kleinbock–Wadleigh, [34]) Let $x \in [0, 1) \setminus \mathbb{Q}$. Then,

(i) $x \in D(\psi)$ if $a_{n+1}(x)a_n(x) \leq \Psi(q_n)/4$ for all sufficiently large n.

(ii) $x \in D(\psi)^c$ if $a_{n+1}(x)a_n(x) > \Psi(q_n)$ for infinitely many n. ∎

As a consequence of this lemma we have the inclusions

$$G(\Psi) \subset D(\psi)^c \subset G(\Psi/4), \tag{7.4}$$

where

$$G(\Psi) := \Big\{x \in [0, 1) : a_n(x)a_{n+1}(x) > \Psi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N}\Big\}.$$

Hussain–Kleinbock–Wadleigh–Wang [27] proved the f-dimensional Hausdorff measure for $G(\Psi)$. For Ψ as defined above and f an essentially sub-linear dimension function, they showed that $\mathcal{H}^f(G(\Psi)) = \infty$ (resp. zero) if $\sum_t tf\left(\frac{1}{t^2\Psi(t)}\right)$ diverges (resp. converges).

Also by Legendre's Theorem it can be easily seen that $G(\Psi)$ contains $\mathcal{K}(3\Psi)$ where

$$\mathcal{K}(\Psi) := \left\{x \in [0, 1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^2\Psi(q)} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

It is worth pointing out that the inclusion

$$\mathcal{K}(3\Psi) \subset G(\Psi) \tag{7.5}$$

along with containment (7.4) was the key observation in proving the divergence part of the Hausdorff measure statement for $G(\Psi)$. Also one can observe that when the sum $\sum_t t\left(\frac{1}{t^2\Psi(t)}\right)^s$ diverges, both the sets $G(\Psi)$ and $\mathcal{K}(3\Psi)$ have full measure. However, since the inclusion (7.5) is proper, it is natural to expect that the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$ is non-trivial. From a measure theoretic point of view there is no new information, however, from a dimension point of view there is more to ask. So the natural question is

How big is the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$?

In Chapter 4, we have answered this question by completely determining the Hausdorff dimension for the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$ for any $C > 0$.

Recall that the set $G(\Psi) \setminus \mathcal{K}(\Psi)$ can be written as

$$\left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Psi(q_n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

In fact we have proved the following result.

Theorem 7.1.5 (Bakhtawar–Bos–Hussian [3]) *Let $\Psi : [1,\infty) \to \mathbb{R}^+$ be a non-decreasing function and $C > 0$. Then*

$$\dim_{\mathrm{H}}\Big(G(\Psi) \setminus \mathcal{K}(C\Psi)\Big) = \frac{2}{\tau+2}, \quad \textit{where } \tau = \liminf_{q\to\infty} \frac{\log \Psi(q)}{\log q}.$$

Now since $\mathcal{K}(3\Psi) \subset G(\Psi)$ and therefore for any non-essentially sub-linear dimension function f, it follows from Theorem **??** that

$$\mathcal{H}^f\big(\mathcal{K}(\Psi)\big) = 0 \quad \text{if} \quad \sum_t t \log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) < \infty.$$

Since this estimate is crude there must be a room for improvement. There are two natural questions here.

Question 7.1.6 Is the convergence estimate given above is best possible?

Question 7.1.7 What is the optimal sum condition so that the f-dimensional Hausdorff measure of $\mathcal{K}(\Psi)$ is infinity for any non-essentially sub-linear dimension function f?

It is plausible that the sum condition will either be the one coming from Theorem 7.1.3 or from the Theorem **??**.

There are a few other important investigations that have been made recently. One of them is the Hausdorff dimension of the level sets,

$$L(\tau) := \left\{ x \in [0,1) : \lim_{n\to\infty} \frac{\log(a_n(x)a_{n+1}(x))}{\log q_n(x)} = \tau \right\},$$

by Huang–Wu [24]. They proved the following dimensional result.

Theorem 7.1.8 (Huang–Wu, [24]) *For any $\tau \geq 0$,*

$$\dim_H(L(\tau)) = \begin{cases} 1 & \text{if} \quad \tau = 0, \\ \frac{2}{\tau+2+\sqrt{\tau^2+4}} & \text{if} \quad \tau > 0. \end{cases}$$

From this theorem it is straightforward to see that $\dim_H(L(\tau))$ as a function of $\tau \in [0,\infty)$, has a jump at $\tau = 0$.

Metrical theory for continued fractions

One of the fundamental result in the metrical theory of continued fraction is Borel–Bernstein's Theorem (1911,1912) which is a kind of Borel–Cantelli 'zero-one' law with respect to the Lebesgue measure.

In this section we will discuss the metrical theory associated with the following set for different m. Consider an arbitrary function $\Phi : \mathbb{N} \to (1,\infty)$ with $\lim_{n\to\infty} \Phi(n) = \infty$, and define

$$\mathcal{E}_m(\Phi) := \left\{ x \in [0,1) : \prod_{i=1}^{m} a_{n+i-1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Case 1. $m = 1$.

In this case, we have

$$\mathcal{E}_1(\Phi) := \{x \in [0,1) : a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

The metrical theory associated with this set has been studied well over the years and a lot is known.

Theorem 7.1.9 (Borel–Bernstein, [7, 9]) *Let $\Phi : \mathbb{N} \to \mathbb{R}^+$ be a positive function. Then*

$$\lambda(\mathcal{E}_1(\Phi)) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \frac{1}{\Phi(n)} < \infty; \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \frac{1}{\Phi(n)} = \infty. \end{cases}$$

Regarding the Hausdorff tmeasure and dimension, some partial results for the Hausdorff dimension of this set were known, for instance Łuczak [39] and Feng–Wu–Tseng [20] determined it for the function $\Phi(n) = a^{b^n}, a > 1, b > 1$. However, a complete Hausdorff dimension result was proven by Wang–Wu [45].

Theorem 7.1.10 (Wang–Wu, [45]) *Let $\Phi : \mathbb{N} \to \mathbb{R}^+$ be an arbitrary positive function. Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \quad \text{and} \quad \log b = \liminf_{n\to\infty} \frac{\log\log \Phi(n)}{n}.$$

Then

$$\dim_{\mathrm{H}} \mathcal{E}_1(\Phi) = \begin{cases} 1; & \text{if } B = 1; \\ \inf\{s \geq 0 : \mathsf{P}(T, -s(\log B + \log|T'|)) \leq 0\} & \text{if } 1 < B < \infty; \\ \frac{1}{1+b} & \text{if } B = \infty. \end{cases}$$

Recall that T is the Gauss map related to the continued fraction expansion, T' denotes the derivative of $T$and P represents the pressure function as defined in Section 2.2.

It is important to mention here that the question of finding the Hausdorff measure of the set $\mathcal{E}_1(\Phi)$ is still open. The set has been substantially generalised to the settings of localised Jarník–Besicovitch set, more details are provided in Section 7.1.

Case 2. $m \geq 2$

In this situation, the first result is due to Kleinbock–Wadleigh who proved the Lebesgue measure result for $m = 2$. Define the set

$$\mathcal{E}_2(\Phi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where $\Phi : \mathbb{N} \to (1, \infty)$ is any function with $\lim_{n\to\infty} \Phi(n) = \infty$.

Theorem 7.1.11 (Kleinbock–Wadleigh, [34]) *Let Φ be a positive function as defined above. Then*

$$\lambda(\mathcal{E}_2(\Phi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)} < \infty; \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)} = \infty. \end{cases}$$

This theorem has been recently (2019) generalised to an arbitrary m by Huang–Wu–Xu [25].

Theorem 7.1.12 (Huang–Wu–Xu, [25]) *Let Φ be as defined above. Then*

$$\lambda(\mathcal{E}_m(\Phi)) = \begin{cases} 0 & \text{if} \quad \sum\limits_{n}^{\infty} \frac{\log^{m-1}\Phi(n)}{\Phi(n)} < \infty; \\ \\ 1 & \text{if} \quad \sum\limits_{n}^{\infty} \frac{\log^{m-1}\Phi(n)}{\Phi(n)} = \infty. \end{cases}$$

Note that $\mathcal{E}_1(\Phi)$ is properly contained in $\mathcal{E}_2(\Phi)$. Since the inclusion is proper, this raises a natural question of the size of the set $\mathcal{F}(\Phi) := \mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi)$. In other words, a natural question is to estimate the size of the set

$$\mathcal{F}(\Phi) = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\},$$

where $\Phi : \mathbb{N} \to (1, \infty)$ be any function with $\lim\limits_{n\to\infty} \Phi(n) = \infty$.

In Chapter 5 we have studied the Hausdorff dimension of $\mathcal{F}(\Phi)$ and proved the following theorem.

Theorem 7.1.13 (Bakhtawar–Bos–Hussain, [2]) *Let $\Phi : \mathbb{N} \to (1, \infty)$ be any function with $\lim\limits_{n\to\infty} \Phi(n) = \infty$ and let $\log B = \liminf\limits_{n\to\infty} \frac{\log \Phi(n)}{n}$ and $\log\, b = \liminf\limits_{n\to\infty} \frac{\log\log \Phi(n)}{n}$. Then*

$$\dim_{\mathrm{H}} \mathcal{F}(\Phi) = \begin{cases} \inf\{s \geq 0 : \mathsf{P}\left(T, -s^2 \log B - s\log(|T'|\right) \leq 0\} & \text{if } 1 < B < \infty; \\ \\ \frac{1}{1+b} & \text{if } B = \infty. \end{cases}$$

Note that if we take $B = 1$ then from the definition of $\mathcal{F}(\Phi)$ we have $a_{n+1}(x) < 1$ which is a contradiction to the assumption that $a_{n+1}(x) \geq 1$. Therefore, B is strictly greater than 1.

Another direction that has been investigated recently is the analogue of the Lagrange's spectrum for the Dirichlet's Theorem instead of the Corollary 1.1.3. Define the Dirichlet spectrum as

$$\mathcal{D}(x) := \sup\left\{ c \geq 1 : \min_{1\leq q<t, p\in\mathbb{Z}} |qx - p| < \frac{1}{ct}, \text{ for all } t \gg 1 \right\}.$$

Clearly any results on this set would improve our understanding on Lagrange's spectrum. From the Lemma 7.1.4, it follows that

$$\mathcal{D}(x) = 1 \iff \limsup_{n\to\infty} a_n(x)a_{n+1}(x) = \infty.$$

This leads naturally to consider the set

$$N\mathcal{D}(\Phi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi(n), \text{ for all } n \geq 1\}.$$

Zhang [47] recently (2020) proved the following result.

Theorem 7.1.14 (Zhang, [47]) *Let $\Phi : \mathbb{N} \to (1, \infty)$ be a function with $\lim_{n\to\infty} \Phi(n) = \infty$. Then*

$$\dim_H N\mathcal{D}(\Phi) = \frac{1}{b+1} \text{ where } \log b = \limsup_{n\to\infty} \frac{\log\log\Phi(n)}{n}.$$

Problem 7.1.15 Determine the Hausdorff dimension of the set $N\mathcal{D}(\Phi)$ if $\log B = \liminf_{n\to\infty} \frac{\log\Phi(n)}{n}$ for any $B \geq 1$. ■

The Hausdorff dimension of the set $\mathcal{E}_m(\Phi)$ for any m has also been established by Huang–Wu–Xu [25].

Theorem 7.1.16 (Huang–Wu–Xu, [25]) *Let $\Phi : \mathbb{N} \to (1, \infty)$ be a function such that $\lim\limits_{n\to\infty} \Phi(n) = \infty$ and*

$$\log B = \liminf_{n\to\infty} \frac{\log\Phi(n)}{n} \text{ and } \log b = \liminf_{n\to\infty} \frac{\log\log\Phi(n)}{n}.$$

Then

$$\dim_{\mathrm{H}} \mathcal{E}_m(\Phi) = \begin{cases} 1, & \text{if } B = 1, \\ \inf\{s \geq 0 : \mathsf{P}\left(T, -f_m(s)\log B - s\log|T'|\right) \leq 0\} & \text{if } 1 < B < \infty; \\ \frac{1}{1+b} & \text{if } B = \infty, \end{cases}$$ ■

where f_m is given by the following iterative formula

$$f_1(s) = s, \quad f_{k+1}(s) = \frac{sf_k(s)}{1 - s + f_k(s)}, \ k \geq 1.$$

Generalised Jarník–Besicovitch set

Recall that an irrationality exponent of an irrational $x \in [0, 1)$ is defined as

$$\vartheta(x) := \sup\left\{\tau : \left|x - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

Moreover for any $\tau \geq 2$, Jarník–Besicovitch Theorem [29, 8] states that

$$\dim_{\mathrm{H}}\{x \in [0, 1) : \vartheta(x) \geq \tau\} = \dim_{\mathrm{H}}\{x \in [0, 1) : \vartheta(x) = \tau\} = \frac{2}{\tau}.$$

Barral–Seuret [4] extended this theorem in the following way. Instead of a constant τ they considered the continuous function $\tau(x)$ and showed that

$$\dim_{\mathrm{H}}\{x \in [0, 1) : \vartheta(x) \geq \tau(x)\} = \dim_{\mathrm{H}}\{x \in [0, 1) : \vartheta(x) = \tau(x)\}$$

$$= \frac{2}{\min\{\tau(x) \;:\; x \in [0,1]\}},$$

where $\tau(x) \geq 2$ is some continuous function on $[0,1]$. They called such a set the localised Jarník–Besicovitch set. This is a fantastic result as it gives the Hausdorff dimension of sets with prescribed irrationality exponent. Their result was further generalised by Wang–Wu–Xu [46] who refashioned the problem in terms of continued fractions and took a dynamical approach.

Recall from Chapter 6, in terms of growth rate of partial quotients Jarník–Besicovitch set can be restated as,

$$J(\tau) := \left\{x \in [0,1) : a_n(x) \geq e^{\left(\frac{\tau-2}{2}\right)S_n(\log|T'(x)|)} \text{ for infinitely many } n \in \mathbb{N}\right\},$$

where $S_n f(x) := f(x) + \cdots + f(T^{n-1}(x))$ represents the ergodic sum of any function f.

In fact, Wang–Wu–Xu [46] introduced the generalised version of $J(\tau)$ as

$$J(\tau; h) := \left\{x \in [0,1) : a_n(x) \geq e^{\tau(x)\cdot S_n h(x)} \text{ for infinitely many } n \in \mathbb{N}\right\},$$

where $h(x)$ and $\tau(x)$ are positive continuous functions defined on $[0,1]$ and calling such points the localised $(\tau; h)$ approximable points. Further, they proved that

$$\dim_{\mathrm{H}} J(\tau, h) = s_{\mathbb{N}}^{(1)} := \inf\{s \geq 0 : \mathsf{P}(T, -s\tau_{\min} h - s\log|T'|) \leq 0\},$$

where $\tau_{\min} = \min\{\tau(x) : x \in [0,1]\}$, P denotes the pressure function and T' is the derivative of Gauss map T.

Keeping in view that the growth rate of partial quotients give us better approximation results, in Chapter 6 we introduce the set of points $x \in [0,1)$ for which the product of an arbitrary block of consecutive partial quotients, in their continued fraction expansion, are growing. To be more precise, for any $r \in \mathbb{N}$ we define the set

$$\mathcal{R}_r(\tau; h) := \left\{x \in [0,1) : \prod_{d=1}^{r} a_{n+d}(x) \geq e^{\tau(x)\cdot S_n h(x)} \text{ for infinitely many } n \in \mathbb{N}\right\}$$

and proved the following theorem.

Theorem 7.1.17 (Bakhtawar, [1]) *Let $h : [0,1] \to (0,\infty)$ and $\tau : [0,1] \to [0,\infty)$ be positive continuous functions with h satisfying the tempered distortion property. Then*

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau; h) = s_{\mathbb{N}}^{(r)} := \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min} h - s\log|T'|) \leq 0\},$$

where $\tau_{\min} = \min\{\tau(x) : x \in [0,1]\}$, $g_1(s) = s$ and $g_r(s) = \frac{s g_{r-1}(s)}{1-s+g_{r-1}(s)}$ for all $r \geq 2$. ■

From Chapter 6, we have noticed this result is more general as it implies various classical results for different values of $\tau(x)$ and $h(x)$.

7.2 Further generalisations and open questions

Recall that for an arbitrary function $\Phi : \mathbb{N} \to (1, \infty)$ with $\lim_{n\to\infty} \Phi(n) = \infty$, we defined the set

$$\mathcal{E}_m(\Phi) = \left\{ x \in [0,1) : \prod_{i=1}^{m} a_{n+i-1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

A natural generalisation and strengthening of the set $\mathcal{E}_m(\Phi)$, as suggested by Kleinbock (in a private communication at the conference *Ergodic Theory, Diophantine Approximation and Related Topics* held at MATRIX, University of Melbourne, Creswick), is to determine the Hausdorff dimension of the following set

$$\mathcal{G}_m^t(\Phi) := \left\{ x \in [0,1) : \prod_{i=1}^{m} a_{n+i-1}^{t_i}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\},$$

where $t_i \in \mathbb{N}$.

This set may be considered as the weighted version of the set $\mathcal{E}_m(\Phi)$ discussed above. We have made substantial progress on this problem (paper is in preparation). It may be possible to obtain some heuristic estimates which may yield the following result. Before we state any results, define

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \text{ and } \log b = \liminf_{n\to\infty} \frac{\log\log \Phi(n)}{n}.$$

Then we may have,

Theorem 7.2.1 *Let B, b be as above. Then*

$$\dim_{\mathrm{H}} \mathcal{G}_m^t(\Phi) = \begin{cases} 1 & \text{if } B = 1; \\ \inf\{s \geq 0 : P(-s\log|T'| - f_{t_1,\cdots,t_m}(s)\log B) \leq 0\} & \text{if } 1 < B < \infty; \\ \frac{1}{1+b} & \text{if } B = \infty, \end{cases}$$

where $f_{t_1,\cdots,t_m}$ is given by the following iterative formula

$$\begin{aligned} f_{t_1,\cdots,t_{\ell+1}}(s) &= \frac{s f_{t_1,\cdots,t_\ell}(s)}{t_{\ell+1} f_{t_1,\cdots,t_\ell}(s) + \max\{0, s-(2s-1)\frac{t_{\ell+1}}{t_i}, 1 \leq i \leq \ell\}} \\ &= \frac{s f_{t_1,\cdots,t_\ell}(s)}{t_{\ell+1} f_{t_1,\cdots,t_\ell}(s) + \max\{0, s-(2s-1)\frac{t_{\ell+1}}{\max_{1\leq i\leq \ell} t_i}\}}, \quad \ell \geq 1 \end{aligned}$$

and $f_{t_1}(s) = \dfrac{s}{t_1}$. ∎

Clearly when exponents of the partial quotients are all identically equal to one, i.e. $t_i = 1, \forall i$, then Theorem 7.2.1 readily implies Theorem 7.1.16.

In fact for m=2 we have obtained the following (paper is in preparation).

Theorem 7.2.2 *Let B, b be as above. Then*

$$\dim_{\mathrm{H}} \mathcal{G}_2^t(\Phi) = \begin{cases} 1 & \text{if } B = 1; \\ \inf\{s \geq 0 : P(-s\log|T'| - f_{t_1,t_2}(s)\log B) \leq 0\} & \text{if } 1 < B < \infty; \\ \frac{1}{1+b} & \text{if } B = \infty, \end{cases}$$

where

$$f_{t_1,t_2}(s) = \frac{s^2}{t_2 s + \frac{1}{t_1}\max\{0, s - (2s-1)\frac{t_2}{t_1}\}}.$$

■

Open questions and problems

In this section we state some of the open problems related to the the Dirichlet improvability which may be worth exploring in the future and which could possibly extended to the inhomogeneous settings.

Improvements to Dirichlet's Theorem in higher dimensions

Similar to the one dimensional ψ-Dirichlet improvable set $D(\psi)$, analogous higher dimensional theory exists. First we state the higher dimensional general form of Dirichlet's Theorem. Let m, n be positive integers and denote by $\mathbb{R}^{mn}$ the space of real $m \times n$ matrices.

Theorem 7.2.3 (Dirichlet's Theorem) *Given any $X \in \mathbb{R}^{mn}$ and $t > 1$, there exist $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that*

$$\|X\mathbf{q} - \mathbf{p}\|^m \leq \frac{1}{t} \text{ and } \|\mathbf{q}\|^n < t.$$

■

Here $\| * \|$ denotes the supremum norm on $\mathbb{R}^i$, $i \in \mathbb{N}$ defined as $\|a\| = \max_{1\leq k\leq i}|a_k|$.

Informally, a matrix X represents a vector valued function $\mathbf{q} \to X\mathbf{q}$ and the above theorem claims that one can pick a not-so-large integer vector $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ so that the output of that function is close to an integer vector. For $m = n = 1$ the theorem just asserts that for any real number x and $t > 1$, one of the first t multiples of x lies within $1/t$ of an integer. Similar to the case $m = n = 1$, Theorem 7.2.3 guarantees a non-trivial integer solution for all t, therefore it is the archetypal uniform Diophantine approximation result. A weaker form of approximation guaranteeing that such a system is solvable for an unbounded set of t, is sometimes known as asymptotic approximation. The following corollary, which follows trivially from this weaker statement, is the archetypal asymptotic approximation result.

Corollary 7.2.4 *For any $X \in \mathbb{R}^{mn}$ there exists infinitely many integer vectors $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$\|X\mathbf{q} - \mathbf{p}\|^m \leq \frac{1}{\|\mathbf{q}\|^n} \quad \textit{for some } \mathbf{p} \in \mathbb{Z}^m.$$

Let $\psi : [t_0, \infty) \to \mathbb{R}^+$ be a non-increasing function where $t_0 > 1$ is fixed. A matrix $X \in \mathbb{R}^{mn}$ is called ψ-Dirichlet if for every sufficiently large t one can find integer vector $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\|X\mathbf{q} - \mathbf{p}\|^m < \psi(t) \text{ and } \|\mathbf{q}\|^n < t.$$

Let $D_{m,n}(\psi)$ denote the set of all ψ-Dirichlet matrices. From now onwards we use the notation $\psi_a(x) := x^{-a}$. Then clearly $D_{1,1}(\psi_1) = \mathbb{R}$, and that for any m, n almost every matrix is ψ_1-Dirichlet. A result in [13] asserts that for $\min(m, n) = 1$ and in [36] for the general case, that for any $c < 1$, the set $D_{m,n}(c\psi_1)$ of $c\psi_1$-Dirichlet matrices has a Lebesgue measure zero. This naturally motivates the following question.

What is a necessary and sufficient condition on a non-increasing function ψ guaranteeing that the set $D_{m,n}(\psi)$ has a zero or full measure? Recall that the answer to this question for $m = n = 1$ has been given in [34].

As discussed earlier in the one dimensional settings, the continued fraction expansions has been useful in characterising the ψ-Dirichlet numbers. However, this machinery is not fully developed in higher dimensions and therefore the following problems are challenging.

Problem 7.2.5 Determine the zero-full law for the set $D_{m,n}(\psi)$ in terms of generalised f-dimensional Hausdorff measure.

However, the Lebesgue measure criterion for the doubly metric analogue of $D_{m,n}(\psi)$ has been recently proved by Kleinbock–Wadleigh [35] by reducing the problem to the shrinking target problem on the space of grids in $\mathbb{R}^{m+n}$. To be precise, let ψ be as above and consider a fixed $\mathbf{b} \in \mathbb{R}^m$. A matrix $X \in \mathbb{R}^{mn}$ is called $\psi_{\mathbf{b}}$-Dirichlet if for every sufficiently large t one can find non-zero integer vector $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|X\mathbf{q} + \mathbf{b} - p\|^m < \psi(t) \quad \text{and} \quad |\mathbf{q}|^n < t. \tag{7.6}$$

Let $D^{\mathbf{b}}_{m,n}(\psi)$ denote the set of all $\psi_{\mathbf{b}}$-Dirichlet matrices. If the inhomogeneous vector $\mathbf{b} \in \mathbb{R}^m$ is not fixed then let $\hat{D}_{m,n}(\psi)$ be the set of all pairs $(X, \mathbf{b}) \in \mathbb{R}^{m+n} \times \mathbb{R}^m$ such that (7.6) holds.

Theorem 7.2.6 (Kleinbock–Wadleigh, [35]) *Given a non-increasing ψ, the set $\hat{D}_{m,n}(\psi)$ has zero (resp. full) Lebesgue measure if and only if the series $\sum_q \frac{1}{q^2\psi(q)}$ diverges (resp. converges).*

Recently (2020), Kim–Kim [33] established the Hausdorff measure analogue of Theorem 7.2.6.

Theorem 7.2.7 (Kim–Kim, [33]) *Let ψ be non-increasing and $0 \leq s \leq mn+m$. Then*

$$\mathcal{H}^s(\hat{D}_{m,n}(\psi)) = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m-s} < \infty; \\ \mathcal{H}^s([0,1]^{mn+m}) & \textit{if} \quad \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m-s} = \infty. \end{cases} \tag{7.7}$$

Recall that for fixed $\mathbf{b} \in \mathbb{R}^m$ the set $D^{\mathbf{b}}_{m,n}(\psi) = \{X \in \mathbb{R}^{mn} : (X, \mathbf{b}) \in \hat{D}_{m,n}(\psi)\}$. In the same article Kim–Kim also provided the Hausdorff measure criterion for the singly metric case.

Theorem 7.2.8 (Kim–Kim, [33]) *Let ψ be non-increasing with $\lim_{t\to\infty}\psi(t) = 0$. Then for any $0 \leq s \leq mn$*

$$\mathcal{H}^s(D^{\mathbf{b}}_{m,n}(\psi)) = \begin{cases} 0 & \textit{if} \quad \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s} < \infty; \\ \mathcal{H}^s([0,1]^{mn}) & \textit{if} \quad \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s} = \infty, \end{cases}$$

for every $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$.

Below we list a few natural problems in this setup.

Problem 7.2.9 Determine the f-Hausdorff measure for $\hat{D}_{m,n}(\psi)$.

Problem 7.2.10 Determine the zero-full law for the set $D^{\mathbf{b}}_{m,n}(\psi)$ in terms of generalised f-dimensional Hausdorff measure.

Problem 7.2.11 What are the precise formulations of the weighted analogues of the sets $D_{m,n}(\psi)$, $D^{\mathbf{b}}_{m,n}(\psi)$ and $\hat{D}_{m,n}(\psi)$?

Given that the proof of Theorem 7.2.6 used the correspondence principle between Diophantine approximation and homogeneous dynamics, the following is a plausible objective.

Complex Numbers System

Moving away from the real setting, to ultrametric settings, improvements to the Dirichlet's Theorem (of the above flavour) are not possible over formal power series or the p-adics. However, solving all of the problems stated above over complex number system will be challenging.

In an analogy with the real case, let $\psi : [t_0, \infty) \to \mathbb{R}^+$ be a non-increasing function with $t_0 \geq 1$ fixed. Consider the set

$$\mathcal{D}_C(\psi) := \left\{ Z \in \mathbb{C} : \begin{array}{l} \exists\, N \text{ such that the system } \left| Z - \dfrac{p_1 + ip_2}{q_1 + iq_2} \right| < \psi(t) \text{ with} \\ |q_1 + iq_2| < t \;\; \text{has a nontrivial integer solution for all } t > N \end{array} \right\}.$$

There has been a lot of progress recently in developing the theory of continued fraction expansion for complex numbers. Hence it is timely to think of the validity of the following problems.

Problem 7.2.12 Characterise any $Z \in \mathcal{D}_C(\psi)$ in terms of its continued fractions.

Problem 7.2.13 Determine the Lebesgue measure of the complex analogue of the set $\hat{D}_{m,n}(\psi)$. ■

Problem 7.2.14 Determine the necessary and sufficient conditions on ψ to guarantee the zero-full law with respect to the f-dimensional Hausdorff measure $\mathcal{H}^f$. ■

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